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Preface

This is a book on numerical methods for singular perturbation problems – in particular, stationary reaction-convection-diffusion problems exhibiting layer behaviour. More precisely, it is devoted to the construction and analysis of layer-adapted meshes underlying these numerical methods.

Numerical methods for singularly perturbed differential equations have been studied since the early 1970s and the research frontier has been constantly expanding since. A comprehensive exposition of the state of the art in the analysis of numerical methods for singular perturbation problems is [141] which was published in 2008. As that monograph covers a big variety of numerical methods, it only contains a rather short introduction to layer-adapted meshes, while the present book is exclusively dedicated to that subject.

An early important contribution towards the optimisation of numerical methods by means of special meshes was made by N.S. Bakhvalov [18] in 1969. His paper spawned a lively discussion in the literature with a number of further meshes being proposed and applied to various singular perturbation problems. However, in the mid 1980s, this development stalled, but was enlivened again by G.I. Shishkin's proposal of piecewise-equidistant meshes in the early 1990s [121, 150]. Because of their very simple structure, they are often much easier to analyse than other meshes, although they give numerical approximations that are inferior to solutions on competing meshes. Shishkin meshes for numerous problems and numerical methods have been studied since and they are still very much in vogue.

With this contribution we try to counter this development and lay the emphasis on more general meshes that – apart from performing better than piecewise-uniform meshes – provide a deeper insight in the course of their analysis.

In this monograph, a classification and a survey are given of layer-adapted meshes for reaction-convection-diffusion problems. The monograph aims at giving a structured and comprehensive account of current ideas in the numerical analysis for various methods on layer-adapted meshes. Both finite differences, finite elements and finite volumes will be covered.

While for finite difference schemes applied to one-dimensional problems, a rather complete convergence theory for arbitrary meshes is developed, the theory is more fragmentary for other methods and problems. They still require the restriction to certain classes of meshes.

The roots of this monograph are a survey lecture presented at the Oberwolfach seminar *Numerical Methods for Singular Perturbation Problems*, 8–14 April 2001 organised by Pieter W. Hemker, Hans-Görg Roos and Martin Stynes, and a review article [91] invited by Thomas J.R. Hughes. I am indebted to their invitations and their continued encouragement.

My thanks also go to a series of colleagues I had the pleasure of working with over the years and who consequently influenced this monograph: Sebastian Franz, Anja Fröhner, R. Bruce Kellogg, Natalia Kopteva, Niall Madden, Hans-Görg Roos, Martin Stynes and Relja Vulanović.

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Notation

u	solution of boundary value problems
u^N	numerical approximations of u
ε, ε_d, ε_c	perturbation parameter(s)
$\mathcal{L}$, $\mathcal{L}^*$	differential operator, its adjoint
L, L^*	discrete operator (discretisation of $\mathcal{L}$), its adjoint
$\mathcal{G}, G$	continuous and discrete Green's functions
Ω	domain
$\partial\Omega = \Gamma$	boundary of Ω
n	outward pointing unit vector normal to $\partial\Omega$
N	number of mesh intervals (in each coordinate direction)
C	generic constant, independent of ε and N
$\bar{\omega}$, $\bar{\omega}_x \times \bar{\omega}_y$	sets of mesh points
h, h_i, $\hbar_i$, k, k_j, $\bar{k}_j$	mesh step sizes
v_x, $v_{\bar{x}}$, $v_{\breve{x}}$, $v_{\hat{x}}$, $v_{\tilde{x}}$, $v_{\check{x}}$	difference operators
u^I	nodal interpolant of u
$\\|\cdot\\|_\infty$	supremum norm
$\\|\cdot\\|_1$	L_1 norm
$\\|\cdot\\|_{-1,\infty}$	$W^{-1,\infty}$ norm
$\|\|\|\cdot\|\|\|_{\varepsilon,\infty}$	ε-weighted $W^{1,\infty}$ norm
$\\|\cdot\\|_{*,\omega}$	discrete version of the norm $\\|\cdot\\|_*$
V^ω	finite element space on the mesh ω
$a(\cdot,\cdot)$	bilinear form
$(\cdot,\cdot)$, $\\|\cdot\\|_0$	scalar product and norm in $L_2(\Omega)$
$\|\cdot\|_1$, $\|\|\|\cdot\|\|\|_\varepsilon$	semi norm and ε-weighted energy norm in $H^1(\Omega)$
$\|\|\|\cdot\|\|\|_{SD}$, $\|\|\|\cdot\|\|\|_\kappa$, $\|\|\|\cdot\|\|\|_\rho$	various method-dependent energy norms
$(\cdot,\cdot)_D$, $\|\cdot\|_{*,D}$, $\\|\cdot\\|_{*,D}$	scalar product and (semi) norm restricted to $D \subset \Omega$
C^l, $C^{l,\alpha}$	(Hölder) function spaces

Contents

Chapter 1
Introduction

Stationary linear reaction-convection-diffusion problems form the subject of this monograph:

$$-\varepsilon u'' - bu' + cu = f \ \text{ in } \ (0,1), \ \ u(0) = \gamma_0, \ u(1) = \gamma_1$$

and its two-dimensional analogue

$$-\varepsilon \Delta u - \boldsymbol{b} \cdot \nabla u + cu = f \ \text{ in } \ \Omega \subset \mathbb{R}^2, \ \ u|_{\partial\Omega} = g$$

with a small positive parameter ε.

Such problems arise in various models of fluid flow [52,53,73]; they appear in the (linearised) Navier-Stokes and in the Oseen equations, in the equations modelling oil extraction from underground reservoirs [32], flows in chemical reactors [3] and convective heat transport with large Péclet number [56]. Other applications include the simulation of semiconductor devices [130].

An Example

Consider the boundary-value problem of finding $u \in C^2(0,1) \cap C[0,1]$ such that

$$-\varepsilon u''(x) - u'(x) = 1 \quad \text{for } x \in (0,1), \quad u(0) = u(1) = 0 \tag{1.1}$$

with $0 < \varepsilon \ll 1$. Formally setting $\varepsilon = 0$, yields

$$-u'(x) = 1 \quad \text{for } x \in (0,1), \quad u(0) = u(1) = 0.$$

Unlike (1.1), this problem does not possess a solution in $C^2(0,1) \cap C[0,1]$. Consequently, when ε approaches zero, the solution of (1.1) is badly behaved in some way.

The solution of (1.1) is

$$u(x,\varepsilon) = \frac{\mathrm{e}^{-1/\varepsilon} - \mathrm{e}^{-x/\varepsilon}}{1 - \mathrm{e}^{-1/\varepsilon}} + 1 - x.$$

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Due to the presence of the exponential $e^{-x/\varepsilon}$, the solution u and its derivatives change rapidly near $x = 0$ for small values of ε. Regions where this happens are referred to as **layers**. Singularly perturbed problems are typically characterised by the presence of such layers. The term **boundary layer** was introduced by Ludwig Prandtl at the Third International Congress of Mathematicians in Heidelberg in 1904.

The solution of (1.1) may be regarded as a function of two variables:

$$u : [0,1] \times (0,1] : (x,\varepsilon) \mapsto u(x,\varepsilon).$$

Taking limits of u for $(x,\varepsilon) \to (0,0)$, we see that

$$\lim_{x\to 0}\lim_{\varepsilon\to 0} u(x,\varepsilon) = 1 \neq 0 = \lim_{\varepsilon\to 0}\lim_{x\to 0} . \tag{1.2}$$

Thus, u as a function of two variables possesses a *classical singularity* at the point $(0,0)$ in the (x,ε)-plane. For this reason we may call (1.1) a singularly perturbed boundary-value problem.

What Is a Singularly Perturbed Problem?

Miller et al. [121] give the following characterisation:

> *The justification for the name 'singular perturbation' is that the nature of the differential equations changes completely in the limit case, when the singular perturbation parameter is equal to zero. For example, ... equations change from being nonlinear parabolic equations to nonlinear hyperbolic equations.*

This describes a phenomenon that *can* lead to the formation of boundary layers and *typically will*—if appropriate boundary conditions are imposed. Roos et al. [141] describe singularly perturbed problems as follows.

> *They are differential equations (ordinary or partial) that depend on a small positive parameter ε and whose solutions (or their derivatives) approach a discontinuous limit as ε approaches zero. Such problems are said to be* singularly perturbed, *where we regard ε as a perturbation parameter.*

Both sources avoid a formal definition:

In the present monograph we propose the following definition.

Definition 1.1. Let B be a function space with norm $\|\cdot\|_B$. Let $D \subset I\!R^d$ be a parameter domain. The continuous function $u : D \to B, \varepsilon \mapsto u(\varepsilon)$ is said to be *regular* for $\varepsilon \to \varepsilon^* \in \partial D$ if there exists a function $u^* \in B$ such that:

$$\lim_{\varepsilon\to\varepsilon^*} \|u_\varepsilon - u^*\|_B = 0,$$

otherwise u_ε is said to be **singular** for $\varepsilon \to \varepsilon^*$.

Let $(\mathcal{P}_\varepsilon)$ be a problem with solution $u(\varepsilon) \in B$ for all $\varepsilon \in D$. We say $(\mathcal{P}_\varepsilon)$ is **singularly perturbed** for $\varepsilon \longrightarrow \varepsilon^* \in \partial D$ in the norm $\|\cdot\|_B$ if u is singular for $\varepsilon \longrightarrow \varepsilon^*$. ♡

Remark 1.2. The definition is norm dependent. For example (1.1), is singularly perturbed in the C^0 norm and the L_∞ norm because of (1.2). However, it is not singularly perturbed in the L_2 norm. There exists a function $u^* : x \mapsto 1 - x$ with

$$\|u_\varepsilon - u^*\|_0 = \mathcal{O}(\varepsilon^{1/2}).$$

The L_2 norm fails to capture the boundary layer in u. ♣

Remark 1.3. Boundary conditions play an important role. Consider the boundary-value problem

$$-\varepsilon u''(x) - u'(x) = 1 \quad \text{for } x \in (0,1), \quad u'(0) = u(1) = 0.$$

This problem is singularly perturbed in the C^1 norm, but it is *not* perturbed in the C^0 norm. The Neumann boundary condition at $x = 0$ leads to the formation of a weak layer only. The first-order derivative remains bounded when $\varepsilon \to 0$. ♣

Uniform Convergence

Classical convergence results for numerical methods for boundary-value problems have the structure

$$\|u - u^h\| \le K h^k,$$

with the maximum mesh size h. The constant K depends on certain derivatives of u and typically tends to infinity as the perturbation parameter ε approaches zero. This means that the maximal step size h has to be chosen proportional to some positive power of ε which is impractical. Therefore, we are looking for so-called *uniform* or *robust* methods where the numerical costs are independent of the perturbation parameter ε. More precisely, we are looking for robust methods in the sense of the following definition:

Definition 1.4. Let u_ε be the solution of a singularly perturbed problem, and let u_ε^N be a numerical approximation of u_ε obtained by a numerical method with N degrees of freedom. The numerical method is said to be **uniformly convergent** or **robust** with respect to the perturbation parameter ε in the norm $\|\cdot\|$ if

$$\|u_\varepsilon - u_\varepsilon^N\| \le \vartheta(N) \text{ for } N \ge N_0$$

with a function ϑ satisfying

$$\lim_{N\to\infty} \vartheta(N) = 0 \text{ and } \partial_\varepsilon \vartheta \equiv 0,$$

and with some threshold value $N_0 > 0$ that is independent of ε. ♡

Scope of the Monograph

Well-developed techniques are available for the computation of solutions outside layers [123, 141], but the problem of resolving layers—which is of great practical importance—is still under investigation. This field has witnessed a stormy development. Layer-adapted meshes have first been proposed by Bakhvalov [18] in the context of reaction-diffusion problems. In the late 1970s and early 1980s, special meshes for convection-diffusion problems were investigated by Gartland [45], Liseikin [113, 114, 116], Vulanović [163–166] and others in order to achieve uniform convergence. The discussion has been livened up by the introduction of special piecewise-uniform meshes by Shishkin [150]. They will be described in more detail in Section 2.1.3. Because of their simple structure, they have attracted much attention and are now widely referred to as Shishkin meshes. A small survey of these meshes can be found in the monograph [141], while [109, 121] and [134] are devoted exclusively to them.

The performance of Shishkin meshes is however inferior to that of Bakhvalov meshes, which has prompted efforts to improve them while retaining some of their simplicity, in particular, the mesh uniformity outside the layers and the choice of mesh transition point where the mesh changes from fine to coarse. For instance, Vulanović [169] uses a piecewise-uniform mesh with more than one transition point. Linß [81, 82] combines the ideas of Bakhvalov and Shishkin, while Beckett and Mackenzie [20] combine an equidistribution idea [31] with a Shishkin-type transition point. With all these various mesh-construction ideas a natural question is:

Can a general theory be derived that allows one to immediately deduce the robust convergence of standard schemes on special meshes and a guaranteed rate of convergence?

A first attempt towards this can be found in [137], where a first-order upwind scheme and a Galerkin FEM are studied on a class of so-called Shishkin-type meshes. A more general criterion was derived in [84, 85] for an upwind-difference scheme in one dimension.

The main purpose of this monograph is to give a survey of recent developments and present the state of the art in the analysis of layer-adapted meshes for a wide range of reaction-convection-diffusion problems.

Chapter 2
Layer-Adapted Meshes

Before surveying a few of the most important ideas from the literature for constructing layer-adapted meshes, we shall introduce some basic concepts for describing layer-adapted meshes.

Throughout $\bar{\omega} : 0 = x_0 < x_1 < \cdots < x_N = 1$ denotes a generic mesh with N subintervals on $[0,1]$, while ω is the set of inner mesh nodes. Set $I_i := [x_{i-1}, x_i]$. The local mesh sizes are $h_i := x_i - x_{i-1}$, $i = 1,\dots,N$, while the maximum step size is $h := \max\limits_{i=1,\dots,N} h_i$.

Definition 2.1. A strictly monotone function $\varphi : [0,1] \to [0,1]$ that maps a uniform mesh $t_i = i/N$, $i = 0,\dots,N$, onto a layer-adapted mesh by $x_i = \varphi(t_i)$, $i = 0,\dots,N$, is called a **mesh generating function.** ♡

A related approach is that of **stretching functions** or **layer-damping transformation** [49,114,115], which are used to transform a problem with layers into a problem whose derivatives are bounded.

For a given mesh generating function $\varphi \in W^{1,1}(0,1)$, the local mesh step sizes can be computed using the formula

$$h_i = \varphi(t_i) - \varphi(t_{i-1}) = \int_{t_{i-1}}^{t_i} \varphi'(t)\mathrm{d}t. \tag{2.1}$$

Another important concept is that of **mesh equidistribution.**

Definition 2.2 (Equidistribution principle). Let $M : [0,1] \to I\!R$ be a positive function a.e. A mesh $\bar{\omega}$ is said to equidistribute the **monitor function** M if

$$\int_{I_i} M(t)\mathrm{d}t = \frac{1}{N}\int_0^1 M(t)\mathrm{d}t \quad \text{for } i = 1,\dots,N.$$

♡

Given a monitor function M the associated mesh generating function is implicitly defined by

$$\int_0^{\varphi(t)} M(s)\mathrm{d}s = \xi \int_0^1 M(s)\mathrm{d}s \quad \text{for } t \in [0,1]$$

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and its derivative by

$$\varphi'(t) = \frac{1}{M(\varphi(t))} \int_0^1 M(s)\mathrm{d}s \quad \text{for } t \in [0,1].$$

2.1 Convection-Diffusion Problems

Consider the boundary-value problem

$$-\varepsilon u'' - bu' + cu = f \quad \text{in } (0,1), \quad u(0) = u(1) = 0, \tag{2.2}$$

where ε is a small positive parameter, $b \geq \beta > 0$ on $[0,1]$. The boundary value problem (2.2) has a unique solution that typically has an exponential boundary layer at $x = 0$ which behaves like $\mathrm{e}^{-\beta x/\varepsilon}$. Figure 2.1 gives a plot of a typical solution

A quantity that will appear frequently in the error estimates later and which characterises the convergence is

$$\vartheta_{cd}^{[p]}(\bar{\omega}) := \max_{i=1,\dots,N} \int_{I_i} \left(1 + \varepsilon^{-1}\mathrm{e}^{-\beta s/p\varepsilon}\right) \mathrm{d}s. \tag{2.3}$$

For example, in Section 4.2 we shall establish for the maximum-norm error of a first-order upwind difference scheme that

$$\left\|u - u^N\right\|_\infty \leq C\vartheta_{cd}^{[1]}(\bar{\omega}) \tag{2.4}$$

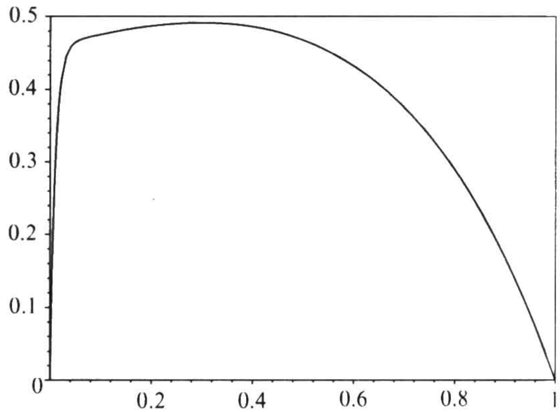

Fig. 2.1 Typical solution of (2.2)

on an arbitrary mesh $\bar{\omega}$. Noting that

$$\int_0^1 \left(1+\varepsilon^{-1}\mathrm{e}^{-\beta s/p\varepsilon}\right) \mathrm{d}s \leq C,$$

we see that an optimal mesh—optimal with respect to the order of convergence—equidistributes the monitor function $M : s \mapsto 1+\varepsilon^{-1}\mathrm{e}^{-\beta s/p\varepsilon}$.

2.1.1 *Bakhvalov Meshes*

Bakhvalov's idea [18] is to use an equidistant t-grid near $x = 0$, then to map this grid back onto the x-axis by means of the (scaled) boundary layer function. That is, grid points x_i near $x = 0$ are defined by

$$q\left(1-\mathrm{e}^{-\beta x_i/\sigma\varepsilon}\right) = t_i = \frac{i}{N} \quad \text{for } i = 0, 1, \dots,$$

where the scaling parameters $q \in (0, 1)$ and $\sigma > 0$ are user chosen: q is roughly the portion of mesh points used to resolve the layer, while σ determines the grading of the mesh inside the layer. Away from the layer, a uniform mesh in x is used with the transition point τ such that, the resulting mesh generating function is $C^1[0, 1]$, i. e.,

$$\varphi(t) = \begin{cases} \chi(t) := -\dfrac{\sigma\varepsilon}{\beta} \ln \dfrac{q-t}{q} & \text{for } t \in [0, \tau], \\ \pi(t) := \chi(\tau) + \chi'(\tau)(t-\tau) & \text{otherwise,} \end{cases}$$

where the point τ satisfies

$$\chi'(\tau) = \frac{1-\chi(\tau)}{1-\tau}. \tag{2.5}$$

Geometrically this means that $(\tau, \chi(\tau))$ is the contact point of the tangent π to χ that passes through the point $(1, 1)$; see Fig. 2.2. When $\sigma\varepsilon \geq \rho q$, the equation (2.5) does not possess a solution. In this case the Bakhvalov mesh is uniform with mesh size N^{-1}.

The nonlinear equation (2.5) cannot be solved explicitly. However, the iteration

$$\tau_0 = 0, \qquad \chi'(\tau_{i+1}) = \frac{1-\chi(\tau_i)}{1-\tau_i}, \quad i = 0, 1, 2 \dots$$