GEOMETRIC TRANSFORMATIONS I

I. M. YAGLOM

translated from the Russian
by ALLEN SHIELDS

Random House
New Mathematical Library

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Allen Shields

University of Michigan

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† The School Mathematics Study Group represented all parts of the mathematical profession and all parts of the country. Its activities were aimed at the improvement of teaching of mathematics in our schools.

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The best way to learn mathematics is to *do* mathematics, and each book includes problems, some of which may require considerable thought. The reader is urged to acquire the habit of reading with paper and pencil in hand; in this way mathematics will become increasingly meaningful to him.

For the authors and editors this is a new venture. They wish to acknowledge the generous help given them by the many high school teachers and students who assisted in the preparation of these monographs. The editors are interested in reactions to the books in this series and hope that readers will write to: Editorial Committee of the NML series, NEW YORK UNIVERSITY, THE COURANT INSTITUTE OF MATHEMATICAL SCIENCES, 251 Mercer Street, New York, N. Y. 10012.

The Editors

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GEOMETRIC TRANSFORMATIONS I

Translator's Preface

The present volume is Part I of *Geometric Transformations* by I. M. Yaglom. The Russian original appeared in three parts; Parts I and II were published in 1955 in one volume of 280 pages. Part III was published in 1956 as a separate volume of 611 pages. In the English translation Parts I and II are published as two separate volumes: NML 8 and NML 21. The School Mathematics Study Group plans to publish a major portion of Part III in yet another volume of the New Mathematical Library. This portion will treat projective geometry and non-Euclidean geometry and will be published later.

In this translation most references to Part III were eliminated, and Yaglom's "Foreword" and "On the Use of This Book" appear, in greatly abbreviated form, under the heading "From the Author's Preface".

This book is not a text in plane geometry. On the contrary, the author assumes that the reader is already familiar with the subject. Most of the material could be read by a bright high school student who has had a term of plane geometry. However, he would have to work; this book, like all good mathematics books, makes considerable demands on the reader.

The book deals with the fundamental transformations of plane geometry, that is, with distance-preserving transformations (translations, rotations, reflections) and thus introduces the reader simply and directly to some important group theoretic concepts.

The relatively short basic text is supplemented by 47 rather difficult problems. The author's concise way of stating these should not discourage the reader; for example, he may find, when he makes a diagram of the given data, that the number of solutions of a given problem depends on the relative lengths of certain distances or on the relative positions of certain given figures. He will be forced to discover for himself the conditions under which a given problem has a unique solution. In the second half of this book, the problems are solved in detail and a discussion of the conditions under which there is no solution, or one solution, or several solutions is included.

The reader should also be aware that the notation used in this book may be somewhat different from the one he is used to. For example, if two lines l and m intersect in a point O, the angle between them is often referred to as $\sphericalangle lOm$; or if A and B are two points, then "the line AB" denotes the line through A and B, while "the line segment AB" denotes the finite segment from A to B.

The footnotes preceded by the usual symbol † were taken over from the Russian version of this book while those preceded by the symbol T have been added in this translation.

I wish to thank Professor Yaglom for his valuable assistance in preparing the American edition of his book. He read the manuscript of the translation and made a number of suggestions. He has expanded and clarified certain passages in the original, and has added several problems. In particular, Problems 4, 14, 24, 42, 43, and 44 in this volume were not present in the original version while Problems 22 and 23 of the Russian original do not appear in the American edition. In the translation of the next part of Yaglom's book, the problem numbers of the American edition do not correspond to those of the Russian edition. I therefore call to the reader's attention that all references in this volume to problems in the sequel carry the problem numbers of the Russian version. However, NML 21 includes a table relating the problem numbers of the Russian version to those in the translation (see p. viii of NML 21).

The translator calls the reader's attention to footnote † on p. 20, which explains an unorthodox use of terminology in this book.

Project for their advice and assistance. Professor H. S. M. Coxeter was particularly helpful with the terminology. Especial thanks are due to Dr. Anneli Lax, the technical editor of the project, for her invaluable assistance, her patience and her tact, and to her assistants Carolyn Stone and Arlys Stritzel.

Allen Shields

From the Author's Preface

This work, consisting of three parts, is devoted to elementary geometry. A vast amount of material has been accumulated in elementary geometry, especially in the nineteenth century. Many beautiful and unexpected theorems were proved about circles, triangles, polygons, etc. Within elementary geometry whole separate "sciences" arose, such as the geometry of the triangle or the geometry of the tetrahedron, having their own, extensive, subject matter, their own problems, and their own methods of solving these problems.

The task of the present work is not to acquaint the reader with a series of theorems that are new to him. It seems to us that what has been said above does not, by itself, justify the appearance of a special monograph devoted to elementary geometry, because most of the theorems of elementary geometry that go beyond the limits of a high school course are merely curiosities that have no special use and lie outside the mainstream of mathematical development. However, in addition to concrete theorems, elementary geometry contains two important general ideas that form the basis of all further development in geometry, and whose importance extends far beyond these broad limits. We have in mind the deductive method and the axiomatic foundation of geometry on the one hand, and geometric transformations and the group-theoretic foundation of geometry on the other. These ideas have been very fruitful; the development of each leads to non-Euclidean geometry. The description of one of these ideas, the idea of the group-theoretic foundation of geometry, is the basic task of this work. . . .

Let us say a few more words about the character of the book. It is intended for a fairly wide class of readers; in such cases it is always necessary to sacrifice the interests of some readers for those of others. The author has sacrificed the interests of the well prepared reader, and has striven for simplicity and clearness rather than for rigor and for logical exactness. Thus, for example, in this book we do not define the general concept of a geometric transformation, since defining terms that are intuitively clear always causes difficulties for inexperienced readers. For the same reason it was necessary to refrain from using directed angles and to postpone to the second chapter the introduction of directed segments, in spite of the disadvantage that certain arguments in the basic text and in the solutions of the problems must, strictly speaking, be considered incomplete (see, for example, the proof on page 50). It seemed to us that in all these cases the well prepared reader could complete the reasoning for himself, and that the lack of rigor would not disturb the less well prepared reader. . . .

The same considerations played a considerable rôle in the choice of terminology. The author became convinced from his own experience as a student that the presence of a large number of unfamiliar terms greatly increases the difficulty of a book, and therefore he has attempted to practice the greatest economy in this respect. In certain cases this has led him to avoid certain terms that would have been convenient, thus sacrificing the interests of the well prepared reader. . . .

The problems provide an opportunity for the reader to see how well he has mastered the theoretical material. He need not solve all the problems in order, but is urged to solve at least one (preferably several) from each group of problems; the book is constructed so that, by proceeding in this manner, the reader will not lose any essential part of the content. After solving (or trying to solve) a problem, he should study the solution given in the back of the book.

The formulation of the problems is not, as a rule; connected with the text of the book; the solutions, on the other hand, use the basic material and apply the transformations to elementary geometry. Special attention is paid to methods rather than to results; thus a particular exercise may appear in several places because the comparison of different methods of solving a problem is always instructive.

There are many problems in construction. In solving these we are not interested in the "simplest" (in some sense) construction—instead the author takes the point of view that these problems present mainly a logical interest and does not concern himself with actually carrying out the construction.

No mention is made of three-dimensional propositions; this restriction does not seriously affect the main ideas of the book. While a section of problems in solid geometry might have added interest, the problems in this book are illustrative and not at all an end in themselves.

The manuscript of the book was prepared by the author at the Orekhovo-Zuevo Pedagogical Institute . . . in connection with the author's work in the geometry section of the seminar in secondary school mathematics at Moscow State University.

I. M. Yaglom

INTRODUCTION

What is Geometry?

On the first page of the high school geometry text by A. P. Kiselyov,[T] immediately after the definitions of *point*, *line*, *surface*, *body*, and the statement "a collection of points, lines, surfaces or bodies, placed in space in the usual manner, is called a geometric figure", the following definition of geometry is given: "*Geometry is the science that studies the properties of geometric figures.*" Thus one has the impression that the question posed in the title to this introduction has already been answered in the high school geometry texts, and that it is not necessary to concern oneself with it further.

But this impression of the simple nature of the problem is mistaken. Kiselyov's definition cannot be called false; however, it is somewhat incomplete. The word "property" has a very general character, and by no means all properties of figures are studied in geometry. Thus, for example, it is of no importance whatever in geometry whether a triangle is drawn on white paper or on the blackboard; the color of the triangle is not a subject of study in geometry. It is true, one might answer, that geometry studies *properties of geometric figures* in the sense of the definition above, and that color is a property of the paper on which the figure is drawn, and is not a property of the figure itself. However, this answer may still leave a certain feeling of dissatisfaction; in order to carry greater conviction one would like to be able to quote a precise "mathematical" definition of exactly which properties of figures are

[T] This is the leading textbook of plane geometry in the Soviet Union.

studied in geometry, and such a definition is lacking. This feeling of dissatisfaction grows when one attempts to explain why it is that, in geometry, one studies the distance from a vertex of a triangle drawn on the board to certain lines, for example, to the opposite side of the triangle, and not to other lines, for example, to the edge of the board. Such an explanation can hardly be given purely on the basis of the definition above.

Before continuing with the presentation we should note that the school textbook cannot be reproached for the incompleteness of its definition. Kiselyov's definition is, perhaps, the only one that can be given at the first stage in the study of geometry. It is enough to say that the history of geometry begins more than 4000 years ago, and the first scientific definition of geometry, the description of which is one of the main aims of this book, was given only about 80 years ago (in 1872) by the German mathematician F. Klein. It required the creation of non-Euclidean geometry by Lobachevsky before mathematicians clearly recognized the need for an exact definition of the subject matter of geometry; only after this did it become clear that the intuitive concept of "geometric figures", which presupposed that there could not be several "geometries", could not provide a sufficient foundation for the extensive structure of the science of geometry.†

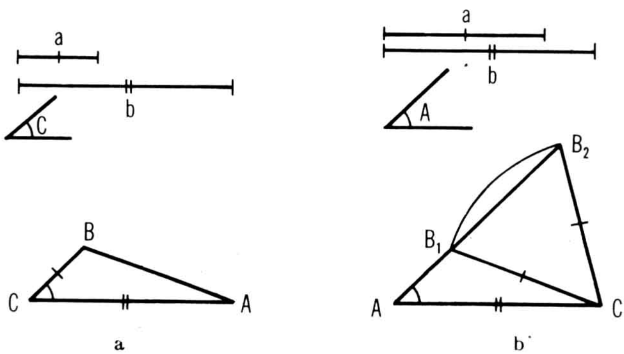

Figure 1

Let us now turn to the clarification of exactly which properties of geometric figures are studied in geometry. We have seen that geometry does not study all properties of figures, but only some of them; before having a precise description of those properties that belong to geometry

† Although non-Euclidean geometry provided the impetus that led to the precise definition of geometry, this definition itself can be fully explained to people who know nothing of the geometry of Lobachevsky.

we can only say that geometry studies "geometric properties" of figures. This addition to Kiselyov's definition does not of itself complete the definition; the question has now become, what are "geometric properties"? and we can answer only that they are "those properties that are studied in geometry". Thus we have gone around in a circle; we defined geometry as the science that studies geometric properties of figures, and geometric properties as being those properties studied in geometry. In order to break this circle we must define "geometric property" without using the word "geometry".

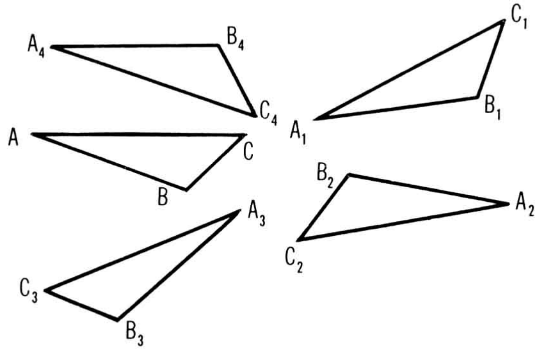

Figure 2

To study the question of what are "geometric properties" of figures, let us recall the following well known proposition: *The problem of constructing a triangle, given two sides a, b, and the included angle C, has only one solution* (Figure 1a).† On second thought, the last phrase may seem to be incorrect; there is really not just one triangle with the given sides a, b, and the included angle C, but there are infinitely many (Figure 2), so that our problem has not just one solution, but infinitely many. What then does the assertion, that there is just one solution, mean?

The assertion that from two sides a, b, and the included angle C *only one* triangle can be constructed clearly means that all triangles having the given sides a, b, and the included angle C are congruent to one another. Therefore it would be more accurate to say that from two sides and the included angle one can construct infinitely many triangles, but they are all congruent to one another. Thus in geometry when one says that there exists a unique triangle having the given sides a, b, and the included angle C, then triangles that differ only in their positions are not

† In contrast to this, the problem of constructing a triangle given the sides a, b, and the angle A opposite one of the given sides can have two solutions (Figure 1b).

considered to be different. And since we defined geometry as the science that studied "geometric properties" of figures, then clearly only figures that have exactly the same geometric properties will be indistinguishable from one another. Thus congruent figures will have exactly the same geometric properties; conversely, figures that are not congruent must have different geometric properties, for otherwise they would be indistinguishable.

Thus we have come to the required definition of geometric properties of figures: *Geometric properties of figures are those properties that are common to all congruent figures.* Now we can give a precise answer to the question of why, for example, the distance from one of the vertices of a triangle to the edge of the board is not studied in geometry: This distance is not a geometric property, since it can be different for congruent triangles. On the other hand, the altitude of a triangle is a geometric property, since corresponding altitudes are always the same for congruent figures.

Now we are much closer to the definition of geometry. We know that geometry studies "geometric properties" of figures, that is, those properties that are the same for congruent figures. It only remains for us to answer the question: "What are congruent figures?"

This last question may disappoint the reader, and may create the impression that thus far we have not achieved anything; we have simply changed one problem into another one, just as difficult. However, this is really not the case; the question of when two figures are congruent is not at all difficult, and Kiselyov's text gives a completely satisfactory answer to it. According to Kiselyov, "*Two geometric figures are said to be congruent if one figure, by being moved in space, can be made to coincide with the second figure so that the two figures coincide in all their parts.*" In other words, congruent figures are those that can be made to coincide by means of a motion; therefore, geometric properties of figures, that is, properties common to all congruent figures, are those properties that are not changed by moving the figures.

Thus we finally come to the following definition of geometry: *Geometry is the science that studies those properties of geometric figures that are not changed by motions of the figures.* For the present we shall stop with this definition; there is still room for further development, but we shall have more to say of this later on.

A nagging critic may not even be satisfied with this definition and may still demand that we define what is meant by a motion. This can be answered in the following manner: *A motion*[T] *is a geometric transformation*

[T] Isometry or rigid motion. From now on the word "isometry" will be used.

of the plane (or of space) carrying each point A into a new point A′ such that the distance between any two points A and B is equal to the distance between the points A′ and B′ into which they are carried.† However, this definition is rather abstract; now that we realize how basic a role isometries play in geometry, we should like to accept them intuitively and then carefully study all their properties. Such a study is the main content of the first volume of this work. At the end of this volume a complete enumeration of all possible isometries of the plane is given, and this can be taken as a new and simpler definition of them. (For more on this see pages 68–70.)

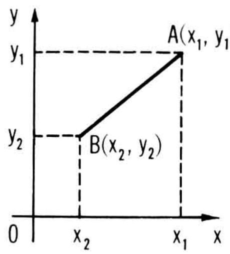

Figure 3

Let us note, moreover, that the study of isometries is essential not only when one wishes to make precise the concepts of geometry, but that it also has a practical importance. The fundamental role of isometries in geometry explains their many applications to the solving of geometric problems, especially construction problems. At the same time the study of isometries provides certain general methods that can be applied to the solution of many geometric problems, and sometimes permits one to combine a series of exercises whose solution by other methods would require

† The distance between two points A and B in the plane is equal to

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

where x_1, y_1 and x_2, y_2 are the coordinates of the points A and B, respectively, in some (it doesn't matter which!) rectangular cartesian coordinate system (Figure 3); thus the concept of distance is reduced to a simple algebraic formula and does not require clarification in what follows.

Analogously, the distance between two points A and B in space is equal to

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

where x_1, y_1, z_1 and x_2, y_2, z_2 are the cartesian coordinates of the points A and B in space.