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Vector Bundles on Curves – New Directions

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Lectures given at the 3rd Session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held in Cetraro (Cosenza), Italy, June 19–27, 1995

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PREFACE

The Third 1995 C.I.M.E. Session:
"Vector Bundles on Curves: New Directions" was held in Grand Hotel San Michele, Cetraro (Cosenza) from 19 to 27 June 1995.

The exciting work of Drinfeld and the recent remarkable insights coming from Theoretical Physics have opened up new directions in the study of vector bundles on curves. The purpose of the session was to give a survey of some of these recent developments.

There were three series of lectures:

1) Kac-Moody Groups, Their Flag Varieties and Moduli Spaces of G-Bundles, by Shrawan Kumar;
2) Drinfeld Shtukas, by G. Laumon

and

3) Drinfeld Modules and Elliptic Sheaves, by U. Stuhler.

The text of the lectures on the third topic was written by U. Stuhler jointly with A. Blum.

M.S. Narasimhan
ICTP

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INFINITE GRASSMANNIANS AND MODULI SPACES OF G–BUNDLES

SHRAWAN KUMAR

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Introduction.

These are notes for my eight lectures given at the C.I.M.E. session on "Vector bundles on curves. New directions" held at Cetraro (Italy) in June 1995. The work presented here was done in collaboration with M.S. Narasimhan and A. Ramanathan and appeared in [KNR]. These notes differ from [KNR] in that we have added three appendices (A)-(C) containing basic definitions and results (we need) on ind-varieties, affine Kac-Moody Lie algebras, the associated groups and their flag varieties. We also have modified the proof (given in §7) of the basic extension result (Proposition 6.5), and we hope that it is more transparent than the one given in [KNR, §7]. We now describe the main result of this note.

Let C be a smooth projective irreducible algebraic curve over $\mathbb{C}$ of any genus and G a connected simply-connected simple affine algebraic group over $\mathbb{C}$. In this note we elucidate the relationship between

(1) the space of vacua ("conformal blocks") defined in Conformal Field Theory, using an integrable highest weight representation of the affine Kac-Moody algebra associated to G and
(2) the space of regular sections ("generalized theta functions") of a line bundle on the moduli space $\mathfrak{M}$ of semistable principal G-bundles on C.

Fix a point p in C and let $\hat{\mathcal{O}}_p$ (resp. $\hat{k}_p$) be the completion of the local ring $\mathcal{O}_p$ of C at p (resp. the quotient field of $\hat{\mathcal{O}}_p$). Let $\mathcal{G} := G(\hat{k}_p)$ (the $\hat{k}_p$-rational points of the algebraic group G) be the loop group of G and let $\mathcal{P} := G(\hat{\mathcal{O}}_p)$ be the standard maximal parahoric subgroup of $\mathcal{G}$. Then the generalized flag variety $X := \mathcal{G}/\mathcal{P}$ is an inductive limit of projective varieties, in fact, of generalized Schubert varieties. One has a basic homogeneous line bundle $\mathfrak{L}(\chi_o)$ on X (cf. §C.6), and the Picard group Pic(X) is isomorphic to $\mathbb{Z}$ which is generated by $\mathfrak{L}(\chi_o)$ (Proposition C.13). There is a central extension $\tilde{\mathcal{G}}$ of $\mathcal{G}$ by the multiplicative group $\mathbb{C}^*$ (cf. §C.4), which acts on the line bundle $\mathfrak{L}(\chi_o)$. By an analogue of the Borel-Weil theorem proved in the Kac-Moody setting by Kumar (and also by Mathieu), the space $H^0(X, \mathfrak{L}(d\chi_o))$ of the regular sections of the line bundle $\mathfrak{L}(d\chi_o) := \mathfrak{L}(\chi_o)^{\otimes d}$ (for any $d \geq 0$) is canonically isomorphic with the full vector space dual $L(\mathbb{C}, d)^*$ of the integrable highest weight (irreducible) module $L(\mathbb{C}, d)$ (with central charge d) of the affine Kac-Moody Lie algebra $\tilde{\mathfrak{g}}$ (cf. §A.2).

Using the fact that any principal G-bundle on $C \setminus p$ is trivial (Proposition 1.3), one sees easily that the set of isomorphism classes of principal G-bundles on C is in bijective correspondence with the double coset space $\Gamma\backslash\mathcal{G}/\mathcal{P}$, where $\Gamma :=$ Mor$(C \setminus p, G)$ is the subgroup of $\mathcal{G}$ consisting of all the algebraic morphisms $C \setminus p \to G$. Moreover, X parametrizes an algebraic family $\mathcal{U}$ of principal G-bundles on C (cf. Proposition 2.8). As an interesting byproduct of this parametrization and rationality of the generalized Schubert varieties, we obtain that the moduli space $\mathfrak{M}$ of semistable principal G-bundles on C is a unirational variety (cf. Corollary 6.3). Now, given a finite dimensional representation V of G, let $\mathcal{U}(V)$ be the family of associated vector bundles on C parametrized by X. We have then the determinant line bundle Det$(\mathcal{U}(V))$ on X, defined as the dual of the determinant of the cohomology of the family $\mathcal{U}(V)$ of vector bundles on C (cf. §3.7). As we mentioned above, Pic (X) is freely generated by the homogeneous line bundle

$\mathfrak{L}(\chi_o)$ on X, in particular, there exists a unique integer m_V (depending on the choice of the representation V) such that $\mathrm{Det}(\mathcal{U}(V)) \simeq \mathfrak{L}(m_V\chi_o)$. We determine this number explicitly in Theorem (5.4), the proof of which makes use of Riemann-Roch theorem. It is shown that the number m_V coincides with the Dynkin index of the representation V. For example, if we take V to be the adjoint representation of G, then $m_V = 2\times$ dual Coxeter number of G (cf. Lemma 5.2 and Remark 5.3). The number m_V is also expressed in terms of the induced map at the third homotopy group level $\pi_3(G) \to \pi_3(SL(V))$ (cf. Corollary 5.6).

The action of Γ on X via left multiplication lifts to an action on the line bundle $\mathfrak{L}(m_V\chi_o)$ (cf. §2.7). Suggested by Conformal Field Theory, we consider the space $H^0(X, \mathfrak{L}(dm_V\chi_o))^\Gamma$ of Γ-invariant regular sections of the line bundle $\mathfrak{L}(dm_V\chi_o)$ (for any $d \geq 0$). This space of invariants is called the space of vacua. More precisely, in Conformal Field Theory, the space of vacua is defined to be the space of invariants of the Lie algebra $\mathfrak{g} \otimes R$ in $L(\mathbb{C}, d)^*$, where R is the ring of regular functions on the affine curve $C \setminus p$ and $\mathfrak{g}$ is the Lie algebra of the group G. We have (by Proposition 6.7) $[L(\mathbb{C}, dm_V)^*]^\Gamma = [L(\mathbb{C}, dm_V)^*]^{\mathfrak{g}\otimes R}$ and, as already mentioned above, $H^0(X, \mathfrak{L}(dm_V\chi_o)) \simeq L(\mathbb{C}, dm_V)^*$. The main result of this note (Theorem 6.6) asserts that (for any $d \geq 0$) the space $H^0(\mathfrak{M}, \Theta(V)^{\otimes d})$ of regular sections of the d-th power of the Θ-bundle $\Theta(V)$ (cf. §3.7) on the moduli space $\mathfrak{M}$ is isomorphic with the space of vacua $[L(\mathbb{C}, dm_V)^*]^\Gamma = [L(\mathbb{C}, dm_V)^*]^{\mathfrak{g}\otimes R}$. Moreover, this isomorphism is canonical up to scalar multiples. This is the connection, alluded to in the beginning of the introduction, between the space of vacua and the space of generalized theta functions. This result has also independently been obtained by Faltings [Fa] and in the case of $G = SL_N$ by Beauville-Laszlo [BL], both by different methods.

We make crucial use of a 'descent' lemma (cf. Proposition 4.1), and an extension result (cf. Proposition 6.5) in the proof of Theorem (6.6). The proof of Proposition (6.5) is given in §7, and relies on the explicit GIT construction of the moduli space of vector bundles.

Our Theorem (6.6) can be generalized to the situation where the curve C has n marked points $\{p_1, \ldots, p_n\}$ together with finite dimensional G-modules $\{V_1, \ldots, V_n\}$ attached to them respectively, by bringing in moduli space of parabolic G-bundles on C.

A purely algebro-geometric study (which does not use loop groups) of generalized theta fuctions on the moduli space of (parabolic) rank two torsion-free sheaves on a nodal curve is made by Narasimhan-Ramadas [NRa]. A factorization theorem and a vanishing theorem for the theta line bundle are proved there. In addition, several other mathematicians (A. Bertram, S. Bradlow, S. Chang, G. Daskalopoulos, B. van Geemen, E. Previato, A. Szenes, M. Thaddeus, R. Wentworth, D. Zagier, $\cdots$) and physicists have studied the space of generalized theta functions (from different view points) in the case when $G = SL(2)$, in the last few years.

Even though we have taken the base field to be the field $\mathbb{C}$ of complex numbers throughout the note, all the results of the note hold good over any algebraically closed field of char 0 (with minor or no modifications in the proofs).

The organization of the note is as follows:

Apart from introducing some notation in §1, we realize the affine flag variety

X as a parameter set for G-bundles. In section (2) we prove that X supports an algebraic family of G-bundles on the curve C (cf. Proposition 2.8). We also realize the group Γ as an ind-group, calculate its Lie algebra, and prove its splitting in this section. Section (3) is devoted to recalling some basic definitions and results on the moduli space of semistable G-bundles, including the definition of the determinant line bundle and the Θ-bundle on the moduli space. We prove a curious result (cf. Proposition 4.1) on algebraic descent in §4. Section (5) is devoted to identifying the determinant line bundle on X with a suitable power of the basic homogeneous line bundle on X. Section (6) contains the statement and the proof of the main result (Theorem 6.6). Finally in Section (7) we prove the basic extension result (Proposition 6.5), using Geometric Invariant Theory. Appendix (A) is devoted to recalling the definition of affine Kac-Moody Lie algebras and its representations. Appendix (B) is an introduction to ind-varieties and ind-groups. Finally in appendix (C), we recall the basic theory of affine Kac-Moody groups and their flag varieties.

1. Affine flag variety as parameter set for G–bundles.

(**1.1**) *Notation.* Throughout the note we take the field $\mathbb{C}$ of complex numbers as the base field. By a scheme we will mean a scheme over $\mathbb{C}$. Let us fix a smooth irreducible projective curve C over $\mathbb{C}$, and a point $p \in C$. Let C^* denote the open set $C \setminus p$. We also fix an affine algebraic connected simply-connected simple group G over $\mathbb{C}$.

For any $\mathbb{C}$-algebra A, by $G(A)$ we mean the A–rational points of the algebraic group G. We fix the following notation to be used throughout the note:

$$\mathcal{G} = \mathcal{G}_{\mathfrak{T}} = G(\hat{k}_p),$$
$$\mathcal{P} = \mathcal{P}_{\mathfrak{T}} = G(\hat{\mathcal{O}}_p), \text{ and}$$
$$\Gamma = \Gamma_{\mathfrak{T}} = G(\mathbb{C}[C^*]),$$

where $\hat{\mathcal{O}}_p$ is the completion of the local ring $\mathcal{O}_p$ of C at p, $\hat{k}_p$ is the quotient field of $\hat{\mathcal{O}}_p$, $\mathbb{C}[C^*]$ is the ring of regular functions on the affine curve C^* (which can canonically be viewed as a subring of $\hat{k}_p$), and $\mathfrak{T}$ is the triple (G, C, p). We will freely use the notation and the results from the three appendices throughout Sections (1)-(7).

We recall the following

(**1.2**) *Definition.* Let H be any (not necessarily reductive) affine algebraic group. By a *principal H-bundle* (for short *H-bundle*) on an algebraic variety X, we mean an algebraic variety E on which H acts algebraically from the right and an H-equivariant morphism $\pi : E \to X$ (where H acts trivially on X), such that π is isotrivial (i.e. locally trivial in the étale topology).

Let H act algebraically on a quasi-projective variety F from the left. We can then form the ***associated bundle with fiber*** F, denoted by $E(F)$. Recall that $E(F)$ is the quotient of $E \times F$ under the H-action given by $g(e, f) = (eg^{-1}, gf)$, for $g \in H$, $e \in E$ and $f \in F$.

Reduction of structure group of E to a closed algebraic subgroup $K \subset H$ is, by definition, a K-bundle E_K such that $E_K(H) \approx E$, where K acts on H by left multiplication. Reduction of structure group to K can canonically be thought of as a section of the associated bundle $E(H/K) \to X$.

Let $\mathcal{X} = \mathcal{X}(H, C)$ denote the set of isomorphism classes of H-bundles on the base C, and $\mathcal{X}_o \subset \mathcal{X}$ denote the subset consisting of those H-bundles on C which are algebraically trivial restricted to C^*. We recall the following proposition essentially due to Harder [H_1, Satz 3.3 and the remark following it].

(1.3) Proposition. *Let H be a connected reductive algebraic group. Then the structure group of any H-bundle on a smooth affine curve Y can be reduced to the connected component $Z^o(H)$ of the centre $Z(H)$ of H.*

In particular, if H as above is semi-simple, then any H-bundle on Y is trivial.

The following map is of basic importance for us in this note. This provides a bridge between the moduli space of G-bundles and the affine (Kac-Moody) flag variety, where G is as in §1.1.

(**1.4**) *Definition* (of the map $\varphi : \mathcal{G} \to \mathcal{X}_o$). Consider the canonical morphisms $i_1 : \mathrm{Spec}\,(\hat{\mathcal{O}}_p) \to C$ and $i_2 : C^* \hookrightarrow C$. Let us take the trivial G-bundles on both the schemes $\mathrm{Spec}\,(\hat{\mathcal{O}}_p)$ and C^*. The fiber product

$$F := \mathrm{Spec}\,(\hat{\mathcal{O}}_p) \underset{C}{\times} C^*$$

of i_1 and i_2 can canonically be identified with $\mathrm{Spec}\,(\hat{k}_p)$. This identification $F \simeq \mathrm{Spec}\,(\hat{k}_p)$ is induced from the natural morphisms

$$\begin{array}{ccccc} & & \mathrm{Spec}\,(\hat{k}_p) & & \\ & \swarrow & & \searrow & \\ \mathrm{Spec}\,(\hat{\mathcal{O}}_p) & & \wr\downarrow & & C^* \\ & \nwarrow & & \nearrow & \\ & & F & & \end{array}$$

By an analogue of "glueing" lemma of Grothendieck ([G, §§2.6, 2.7], [BL_2]), to give a G-bundle on C, it suffices to give an automorphism of the trivial G-bundle on $\mathrm{Spec}\,(\hat{k}_p)$, i.e., to give an element of $\mathcal{G} := G(\hat{k}_p)$. (Observe that since we have a cover of C by only two schemes, the cocycle condition is vacuously satisfied.) This is, by definition, the map $\varphi : \mathcal{G} \to \mathcal{X}_o$.

(1.5) Proposition. *The map φ (defined above) factors through the double coset space to give a bijective map (denoted by)*

$$\bar{\varphi} : \Gamma \backslash \mathcal{G} / \mathcal{P} \to \mathcal{X}_o\,.$$

(Observe that, by Proposition (1.3), $\mathcal{X}_o = \mathcal{X}$ since G is assumed to be connected and semi-simple.)

Proof. From the above construction, it is clear that for $g, g' \in \mathcal{G}$, $\varphi(g)$ is isomorphic with $\varphi(g')$ (written $\varphi(g) \approx \varphi(g')$) if and only if there exist two G-bundl isomorphisms :

$$\begin{array}{ccc} \operatorname{Spec}(\widehat{\mathcal{O}}_p)\times G & \xrightarrow[\sim]{\theta_1} & \operatorname{Spec}(\widehat{\mathcal{O}}_p)\times G \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec}(\widehat{\mathcal{O}}_p) & \end{array}$$

and

$$\begin{array}{ccc} C^*\times G & \xrightarrow[\sim]{\theta_2} & C^*\times G \\ & \searrow \quad \swarrow & \\ & C^* & \end{array}$$

such that the following diagram is commutative:

$$(*)\qquad \begin{array}{ccc} \operatorname{Spec}(\hat{k}_p)\times G & \xrightarrow{\theta_1|\operatorname{Spec}(\hat{k}_p)} & \operatorname{Spec}(\hat{k}_p)\times G \\ \Big\downarrow g' & & \Big\downarrow g \\ \operatorname{Spec}(\hat{k}_p)\times G & \xrightarrow{\theta_2|\operatorname{Spec}(\hat{k}_p)} & \operatorname{Spec}(\hat{k}_p)\times G\,. \end{array}$$

Any G-bundle isomorphism θ_1 (resp. θ_2) as above is given by an element $h\in\mathcal{P}$ (resp. $\gamma\in\Gamma$). In particular, from the commutativity of the above diagram $(*)$, $\varphi(g)\approx\varphi(g')$ if and only if there exist $h\in\mathcal{P}$ and $\gamma\in\Gamma$ such that $gh=\gamma g'$, i.e., $\gamma^{-1}gh=g'$. This shows that the map φ factors through $\Gamma\backslash\mathcal{G}/\mathcal{P}$ to give an injective map $\bar{\varphi}$. The surjectivity of $\bar{\varphi}$ follows immediately from the definition of $\mathcal{X}_o$, and the fact that any G-bundle on $\operatorname{Spec}(\widehat{\mathcal{O}}_p)$ is trivial. □

(1.6) *Remark.* $\mathcal{G}/\mathcal{P}$ should be thought of as a parameter space for G-bundles E together with a trivialization of $E_{|C^*}$ (cf. Proposition 2.8).

2. Affine flag variety parametrizing an algebraic family and realizing Γ as an ind-group.

Recall the definition of the group $\Gamma\subset\mathcal{G}$ from §1.1.

(2.1) Lemma. *The group* Γ *is an ind-group.*

Proof.[1] Embed $G\hookrightarrow SL_N\subset M_N$, where M_N is the space of $N\times N$ matrices over $\mathbb{C}$. This induces an injective map $i:\Gamma\hookrightarrow \operatorname{Mor}(C^*,M_N)$, where Mor (C^*,M_N) denotes the set of all the morphisms from C^* to M_N . Take a $\mathbb{C}$–basis $\{f_1,f_2,f_3,\cdots\}$ of $\mathbb{C}[C^*]$ (the ring of regular functions) such that $\operatorname{ord}_p f_n\le\operatorname{ord}_p f_{n+1}$ for any $n\ge1$, where $\operatorname{ord}_p f_n$ denotes the order of the pole of f_n at p. The set Mor (C^*,M_N) has a filtration $\operatorname{Mor}_0\subseteq\cdots\subseteq\operatorname{Mor}_n\subseteq\ldots$, where Mor_n is the (finite dimensional) vector

[1] I thank R. Hammack for some simplification in my original argument.

space of all those morphisms $\theta : C^* \to M_N$ such that all its matrix entries have poles of order $\leq n$. Set $\Gamma_n = i^{-1}(\mathrm{Mor}_n)$. Any $\theta = (\theta_{i,j}) \in \mathrm{Mor}_n$ can be written as $\theta_{i,j} = \sum_{k=1}^{k(n)} z^k_{i,j} f_k$ (for some $k(n)$). We take $(z^k_{i,j})$ as the coordinates on Mor_n. It is easy to see that $\Gamma_n \hookrightarrow \mathrm{Mor}_n$ is given by the vanishing of some polynomials in $(z^k_{i,j})$, in particular, Γ_n is a closed subvariety of the affine space Mor_n. (We put the reduced structure on Γ_n.) This gives rise to the ind-variety structure on Γ as a closed ind-subvariety of Mor (C^*, M_N). It is easy to see (from the definition of the ind-variety structure on Γ) that Γ in fact is an ind-group. Moreover, this ind-variety structure on Γ does not depend upon the particular choice of the embedding $G \hookrightarrow SL_N$. □

The following lemma determines the Lie algebra of the ind-group Γ.

(2.2) Lemma. *The Lie algebra* $Lie\,\Gamma$ *is isomorphic with* $\mathfrak{g} \otimes_{\mathbb{C}} R$, *where* $\mathfrak{g} :=$ *Lie* G, $R := \mathbb{C}[C^*]$, *and the bracket in* $\mathfrak{g}\otimes R$ *is defined as* $[X\otimes p, Y\otimes q] = [X,Y]\otimes pq$, *for* $X, Y \in \mathfrak{g}$ *and* $p, q \in R$. *The isomorphism* $Lie\,\Gamma \simeq \mathfrak{g} \otimes R$ *is obtained by considering the differential of the evaluation map at each point of* C^*.

Proof. Choose an embedding $G \hookrightarrow SL_N \subset M_N$ as in the proof of Lemma (2.1). This gives rise to a closed immersion $i : \Gamma \hookrightarrow$ Mor (C^*, M_N). In particular, it induces an injective map $di : T_e(\Gamma) = \mathrm{Lie}\,\Gamma \hookrightarrow T_I(\mathrm{Mor}) \simeq$ Mor at the Zariski tangent space level (where I is the identity matrix and Mor = Mor (C^*, M_N)). We claim that di is a Lie algebra homomorphism, if we endow Mor $\simeq M_N(R)$ with the standard Lie algebra structure, where $M_N(R)$ is the space of $N \times N$ matrices over R. To prove this, consider the following commutative diagram (for any fixed $x \in C^*$):

$$\begin{array}{ccc} T_e(\Gamma) & \overset{di}{\hookrightarrow} & M_N(R) \\ \downarrow & & \downarrow \\ \mathfrak{g} = T_e(G) & \hookrightarrow & M_N\ , \end{array}$$

where the vertical maps are induced by the evaluation map $e_x : R \to \mathbb{C}$ given by $p \mapsto p(x)$. Since the bottom horizontal map is a Lie algebra homomorphism, and so are the vertical maps, we obtain that di itself is a Lie algebra homomorphism. It is further clear, from the above commutative diagram, that the image of di is contained in $\mathfrak{g} \otimes R$, where $\mathfrak{g}$ is identified with its image in M_N.

Next, we prove that the image of di contains at least the set $\mathfrak{g} \otimes R$:

Fix any vector $X \in \mathfrak{g} \subset M_N$ such that X is a nilpotent matrix and take $p \in R$, and define a morphism $\mathbf{A}^1 \to \Gamma$ by $z \mapsto exp(zX \otimes p)$. (Since X is nilpotent, the image is indeed contained in Γ.) It is easy to see that the image of the induced map (at the tangent space level at 0) is precisely the space $\mathbb{C}(X \otimes p)$. But since the nilpotent matrices $X \in \mathfrak{g}$ span $\mathfrak{g}$, the assertion follows. This completes the proof of the lemma. □

We prove the following interesting lemma (even though we do not make use of it).

(2.3) Lemma. *Let Y be a connected variety (over $\mathbb{C}$). Then any regular map $Y \to \mathbb{C}^*$, which is null-homotopic in the topological category, is a constant.*

(Observe that if the singular cohomology $H^1(Y, \mathbb{Z}) = 0$, then any continuous map $Y \to \mathbb{C}^*$ is null-homotopic.)

Proof. Assume, if possible, that there exists a null-homotopic non-constant regular map $\lambda : Y \to \mathbb{C}^*$. Since λ is algebraic, there exists a number $N > 0$ such that the number of irreducible components of $\lambda^{-1}(z) \leq N$, for all $z \in \mathbb{C}^*$. Now we consider the N'-sheeted covering $\pi_{N'} : \mathbb{C}^* \to \mathbb{C}^* (z \mapsto z^{N'})$, for any $N' > N$. Since λ is null-homotopic, there exists a (regular) lift $\tilde{\lambda} : Y \to \mathbb{C}^*$ (cf. [Se$_1$, Proposition 20]), making the following diagram commutative:

$$\begin{array}{ccc} & & \mathbb{C}^* \\ & \overset{\tilde{\lambda}}{\nearrow} & \downarrow{\scriptstyle \pi_{N'}} \\ Y & \underset{\lambda}{\longrightarrow} & \mathbb{C}^* . \end{array}$$

Since $\tilde{\lambda}$ is regular and non-constant, by Chevalley's theorem, Im $\tilde{\lambda}$ (being a constructible set) misses only finitely many points of $\mathbb{C}^*$. In particular, there exists a $z_o \in \mathbb{C}^*$ (in fact a Zariski-open set of points) such that $\pi_{N'}^{-1}(z_o) \subset \operatorname{Im} \tilde{\lambda}$. But then the number of irreducible components of $\lambda^{-1}(z_o) = \tilde{\lambda}^{-1}(\pi_{N'}^{-1})(z_o) \geq N' > N$, a contradiction to the choice of N. This proves the lemma. □

We will use the following proposition in the proof of assertion (c) contained in the proof of Theorem (6.6).

(2.4) Proposition. *There does not exist any non-constant regular map $\lambda : \Gamma \to \mathbb{C}^*$.*

Proof. Fix a Borel subgroup $B \subset G$ and let U be its unipotent radical. Fix any $g \in G$. Consider the subgroup $\operatorname{Mor}(C^*, gUg^{-1}) \subset \Gamma$ consisting of all the regular maps $f : C^* \to gUg^{-1}$. We put the ind-group structure on $\operatorname{Mor}(C^*, gUg^{-1})$ similar to that of Γ as in the proof of Lemma (2.1). We denote the inclusion (which is a regular map) by

$$\theta = \theta_g : \operatorname{Mor}(C^*, gUg^{-1}) \hookrightarrow \Gamma .$$

Let $\lambda : \Gamma \to \mathbb{C}^*$ be a regular map, and consider the regular map

$$\lambda \circ \theta : \operatorname{Mor}(C^*, gUg^{-1}) \to \mathbb{C}^* .$$

The exponential map induces an isomorphism of the ind-varieties $\operatorname{Mor}(C^*, gUg^{-1}) \approx \operatorname{Mor}(C^*, \mathfrak{n}) = \mathfrak{n} \otimes_{\mathbb{C}} \mathbb{C}[C^*]$, where $\mathfrak{n} :=$ Lie U. In particular, $\operatorname{Mor}(C^*, gUg^{-1})$ is an inductive limit of (finite dimensional) affine spaces and hence the regular map $\lambda \circ \theta$ is constant. So the derivative map at the tangent level $d(\lambda \circ \theta) : T_e(\operatorname{Mor}(C^*, gUg^{-1})) \to T_{\lambda(e)}(\mathbb{C}^*)$ is the zero map.

As seems to be well known, the group Γ is connected. I do not know to whom this result should be attributed, but there is an interesting proof of this due to Drinfeld.

Now assume (if possible) that λ is non-constant. Then (using connectedness of Γ) there exists a positive integer n and a point $h \in \Gamma_n$ such that the derivative map $d(\lambda_{|\Gamma_n}) : T_h(\Gamma_n) \to T_{\lambda(h)}(\mathbb{C}^*)$ is non-zero (where Γ_n is the filtration of Γ as in the proof of Lemma 2.1). In particular, the derivative map $(d\lambda)_h : T_h(\Gamma) \to T_{\lambda(h)}(\mathbb{C}^*)$ is non-zero. By translating the map λ, if necessary, we can assume that $h = e$. But since $T_e(\operatorname{Mor}(C^*, gUg^{-1})) = gng^{-1} \otimes_{\mathbb{C}} \mathbb{C}[C^*]$ (by the same proof as of Lemma 2.2), we obtain that $(d\lambda)_e$ vanishes on the sum $\mathfrak{s} := \sum_{g\in G}(gng^{-1}) \otimes_{\mathbb{C}} \mathbb{C}[C^*]$. Further $\sum_{g\in G} gng^{-1} = \mathfrak{g}$ and hence $T_e(\Gamma) = \mathfrak{s}$ (by Lemma 2.2). In particular, $(d\lambda)_e$ vanishes on the whole tangent space $T_e(\Gamma)$, a contradiction! This proves that the map λ is constant on Γ, proving the proposition. □

Remark. Simple-connectedness of Γ of course will imply the above proposition (in view of Lemma 2.3). In fact, it is very likely that the space Γ is homotopically equivalent to the corresponding space Γ_{top} consisting of all the continuous maps $C^* \to G$ under the compact-open topology. This of course will give the connected and simple-connectedness of Γ, by using a result of Thom [GK, Theorem 5.10]. A student of mine R. Hammack is trying to give a proof of this homotopy equivalence by using some ideas similar to [PS, Proof of Proposition 8.11.6], albeit in the algebraic category, together with a variant of a result of Hurtubise [Hur, Theorem 1.3].

Recall from Proposition (C.12) that $X = X_{\text{rep}} = X_{\text{lat}}$ is a projective ind-variety.

(2.5) Lemma. *The left multiplication of* Γ *on* X *is a morphism* $\delta : \Gamma \times X \to X$.

Proof. We will consider the X_{lat} description of X (cf. §C.9). It is clear that for any non-negative integers m, n , $\delta(\Gamma_n \times \hat{X}_m) \subseteq \hat{X}_{k(n,m)}$, for some $k(n,m)$. Now from the explicit description of the variety structures (on Γ and X_{lat}), it is easy to check that $\delta_{n,m} := \delta_{|\Gamma_n \times \hat{X}_m}$ is a morphism.

This proves the lemma. □

Restrict the central extension (1) of §(C.4) to get a central extension

$$(1) \qquad 1 \to \mathbb{C}^* \to \tilde{\Gamma} \underset{\pi}{\to} \Gamma \to 1,$$

where $\tilde{\Gamma}$ is by definition $\pi^{-1}(\Gamma)$.

(2.6) Splitting of the central extension over Γ (SL_N case). The basic reference for this subsection is [PS, §7.7]. We first consider the case of $G = SL_N$ and follow the same notation as in §(C.7). In particular, $\mathcal{G}^o := SL_N(\mathbb{C}((t)))$, $\mathcal{P}^o = SL_N(\mathbb{C}[[t]])$, $X^o = \mathcal{G}^o/\mathcal{P}^o$, $V = \mathbb{C}^N$, $V((t)) = V \otimes_{\mathbb{C}} \mathbb{C}((t))$, and $L_o = V \otimes_{\mathbb{C}} \mathbb{C}[[t]]$. Let $GL(W)$ denote the group of $\mathbb{C}$-linear isomorphisms of a vector space W.

Define the subgroup $\mathcal{H}$ of $\mathcal{G}^o \times GL(L_o)$ by

$$\mathcal{H} = \{(g, E) \in \mathcal{G}^o \times GL(L_o) : g^+ - E : L_o \to L_o \text{ has finite rank}\},$$

where $g = \begin{pmatrix} g^+ & * \\ * & * \end{pmatrix}$ with respect to the decomposition

$$V((t)) = L_o \oplus (V \underset{\mathbb{C}}{\otimes} t^{-1}\mathbb{C}[t^{-1}]). \tag{1}$$

Let $\mathcal{N} \subset \mathcal{H}$ be the normal subgroup defined as $\mathcal{N} = \{(1, E) \in \mathcal{H} : \det E = 1\}$. (Observe that since $I - E : L_o \to L_o$ has finite rank, i.e., has finite dimensional image, the determinant of E is well defined.)

It is not difficult to see that the projection on the first factor gives rise to a central extension:

$$1 \to \mathbb{C}^* \to \mathcal{H}/\mathcal{N} \to \mathcal{G}^o \to 1 . \tag{2}$$

We now give an alternative description of the line bundle $\mathfrak{L}(\chi_o)$ on X^o (cf. §C.6):

Recall the definition of the set $\mathcal{F}$ and the map $\beta : X^o \to \mathcal{F}$ from §(C.7). For any $W \in \mathcal{F}$, define S_W as the set of $\mathbb{C}$-linear isomorphisms $\theta : L_o \to W$ such that $\pi_1\theta - I : L_o \to L_o$ has finite rank, where $\pi_1 : V((t)) \to L_o$ is the projection on the L_o factor with respect to the decomposition (1).

Define the vector space $\mathcal{V}_W$ over $\mathbb{C}$ with basis parametrized by the set S_W, i.e., an element of $\mathcal{V}_W$ is of the form $\sum_{\theta \in S_W} z_\theta \theta$, where all but finitely many $z_\theta \in \mathbb{C}$ are zero. Let $\mathcal{V}'_W \subset \mathcal{V}_W$ be the subspace spanned by $\{\theta - \det(\theta'^{-1}\theta)\theta'\}_{\theta,\theta' \in S_W}$ and let $\mathcal{L}_W = \mathcal{V}_W/\mathcal{V}'_W$. (Note that $\theta'^{-1}\theta - I$ has finite rank as an endomorphism of L_o and hence $\det(\theta'^{-1}\theta)$ is well defined.) Then $\mathcal{L}_W$ is a 1-dimensional vector space. Now define the line bundle $\mathcal{L} \xrightarrow{\eta} \mathcal{F}$, where $\eta^{-1}(W) = \mathcal{L}_W$ for any $W \in \mathcal{F}$. As proved in [PS, §7.7], the line bundle $\mathcal{L}$ is an algebraic line bundle on $\mathcal{F}$ (with respect to the ind-variety structure on $\mathcal{F}$ as in §C.7). It is easy to see that $\mathcal{L}_{|\mathcal{F}_1}$ is the restriction of the basic (negative ample) line bundle on $Gr(N, 2N)$ under the identification $\mathcal{F}_1 \xrightarrow{\sim} Gr(N, 2N)^{1+\bar{t}_1}$ (cf. §C.7). Let $\mathcal{L}_o$ be the pull-back of the line bundle $\mathcal{L}$ to X^o via the isomorphism $\beta : X^o \xrightarrow{\sim} \mathcal{F}$. In view of Proposition (C.13), it is easy to see that the dual line bundle $\mathcal{L}_o^*$ is isomorphic with the line bundle $\mathfrak{L}(\chi_o)$.

Now we define an action α of the group $\mathcal{H}/\mathcal{N}$ on $\mathcal{L}$ as follows: For $(g, E) \in \mathcal{H}$, define

$$\alpha(g, E)[z, \theta]_W = [z, g\theta E^{-1}]_{gW},$$

where for $z \in \mathbb{C}$ and $\theta \in S_W$, $[z, \theta]_W$ denotes the equivalence class of $z\theta$. This action factors through an action of $\mathcal{H}/\mathcal{N}$ and moreover for any fixed $(g, E) \in \mathcal{H}$, $\alpha(g, E)$ is an algebraic automorphism of the line bundle $\mathcal{L}$ (and hence of $\mathcal{L}_o$) inducing the map L_g on the base (cf. §C.6). Using this, the group $\mathcal{H}/\mathcal{N}$ can canonically be identified with the Mumford group $\mathrm{Aut}(\mathfrak{L}_o)$ defined in §(C.6). In particular, the central extension $\mathcal{H}/\mathcal{N}$ is isomorphic with $\tilde{\mathcal{G}}$.

Finally we construct a splitting of $\mathcal{H}/\mathcal{N}$ over Γ as follows:

Choose an element $g_o \in \mathcal{G}^o$ such that the associated rank-N vector bundle $\tilde{\varphi}(g_o)$ on C twisted by $\mathcal{O}((g-1)p), E := \tilde{\varphi}(g_o)((g-1)p)$ (where g is the genus of the curve C) has all its cohomology 0. Then considering the local cohomology sequence (for the curve C with support in p) with coefficients in the vector bundle E, we deduce that

$$V((t)) = L_o \oplus t^{1-g} g_o^{-1}(V \otimes_{\mathbb{C}} \mathbb{C}[C^*]) , \tag{3}$$

where $V \otimes_{\mathbb{C}} \mathbb{C}[C^*]$ is identified as a subspace of $V((t))$ by choosing a parameter t around $p \in C$.

We first construct the splitting of $\mathcal{H}/\mathcal{N}$ over $\Gamma_{g_o} := g_o^{-1}\Gamma g_o$. Define the group homomorphism $\sigma_{g_o} : \Gamma_{g_o} \to \mathcal{H}$ by $\sigma_{g_o}(\gamma) = (\gamma, \gamma^{+'})$, where $\gamma = \begin{pmatrix} \gamma^{+'} & 0 \\ * & * \end{pmatrix}$ with respect to the decomposition (3). (Observe that Γ_{g_o} keeps the second factor stable and hence $\gamma^{+'} \in GL(L_o)$.) The group homomorphism $\overline{\sigma}_{g_o} : \Gamma_{g_o} \to \mathcal{H}/\mathcal{N}$ (where $\overline{\sigma}_{g_o}$ is the map σ_{g_o} followed by the canonical map $\mathcal{H} \to \mathcal{H}/\mathcal{N}$) splits the central extension (2) over Γ_{g_o}. Now take any preimage $\overline{g}_o$ of g_o in $\mathcal{H}/\mathcal{N}$, and define the splitting $\overline{\sigma} : \Gamma \to \mathcal{H}/\mathcal{N}$ $(\gamma \mapsto \overline{g}_o\overline{\sigma}_{g_o}(g_o^{-1}\gamma g_o)\overline{g}_o^{-1})$.

Since $\mathcal{H}/\mathcal{N}$ acts on the line bundle $\mathcal{L}_o$, so is Γ (via the homomorphism $\overline{\sigma}$). It can be easily seen that the action $\Gamma \times \mathcal{L}_o \to \mathcal{L}_o$ is a morphism of ind-varieties. Moreover, let $\overline{\sigma}' : \Gamma \to \mathcal{H}/\mathcal{N}$ be another splitting of Γ such that the induced action $\Gamma \times \mathcal{L}_o \to \mathcal{L}_o$ is again a morphism of ind-varieties. Then we claim that $\overline{\sigma}' = \overline{\sigma}$: There is a group homomorphism $\alpha : \Gamma \to \mathbb{C}^*$ such that (cf.(2)) $\overline{\sigma}' = \alpha\overline{\sigma}$. Further α is a morphism of ind-varieties (since the action of Γ on $\mathcal{L}_o$ in both the cases is regular). But then α is identically 1 (cf. Proposition 2.4, see also Remark 6.8(c)). This proves the uniqueness of such a splitting.

Since the line bundle $\mathcal{L}_o$ is isomorphic with the homogeneous line bundle $\mathfrak{L}(-\chi_o)$, it is easy to see that the group Γ acts morphically on the representation $L(\mathbb{C}, 1)$ and hence on any $L(\mathbb{C}, d)$ (for $d \geq 0$, where $L(\mathbb{C}, d)$ is the irreducible representation of the affine Lie algebra $\tilde{sl}_N$ with central charge d, cf. §A.2).

(2.7) Splitting of the central extension over Γ (general case). We now come to the case of general G as in §1.1. Take a finite dimensional representation V of G such that the group homomorphism $\gamma : G \to \mathrm{SL}(V)$ has finite kernel, and consider the induced map at the Lie algebra level $d\gamma : \mathfrak{g} \to sl(V)$, where $sl(V)$ is the Lie algebra of trace 0 endomorphisms of V. We denote the Lie algebra $sl(V)$ by $\mathfrak{g}^o$. The Lie algebra homomorphism $d\gamma$ induces a Lie algebra homomorphism $\tilde{\gamma} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}^o$ defined by (cf. §A.1)

$$X \otimes p \mapsto (d\gamma(X)) \otimes p \ , \ \text{and } K \mapsto m_V K^o \ ,$$

for $X \in \mathfrak{g}$ and $p \in \mathbb{C}[t^{\pm 1}]$; where K (resp. K^o) is the canonical central element of $\tilde{\mathfrak{g}}$ (resp. $\tilde{\mathfrak{g}}^o$), and m_V is the Dynkin index of the representation V (cf. §5.1).

To distinguish the objects corresponding to $\mathrm{SL}(V)$ from that of G, we denote the former by a superscript o. Let us consider the irreducible representation $L^o(\mathbb{C}, 1)$ for the Lie algebra $\tilde{\mathfrak{g}}^o$ with central charge 1 and restrict it to the Lie algebra $\tilde{\mathfrak{g}}$ via the homomorphism $\tilde{\gamma}$. It can be seen that the $\tilde{\mathfrak{g}}$-submodule of $L^o(\mathbb{C}, 1)$ generated by the highest weight vector v_o is isomorphic with $L(\mathbb{C}, m_V)$.

The representation γ also gives rise to a morphism of the corresponding affine flag varieties $\hat{\gamma} : X \to X^o$, and a morphism of ind-groups $\Gamma \to \Gamma^o$. It is easy to see that the basic homogeneous line bundle $\mathfrak{L}^o(\chi_o)$ on X^o pulls-back to the line bundle $\mathfrak{L}(m_V\chi_o)$ on X. In particular, the group Γ acts morphically on the line bundles $\mathfrak{L}(dm_V\chi_o)$ (for any $d \in \mathbb{Z}$) and hence Γ also acts morphically on the representation space $L(\mathbb{C}, dm_V)$.

We come now to the following proposition, asserting that $X = \mathcal{G}/\mathcal{P}$ supports an algebraic family.

(2.8) Proposition. *(a) There is an algebraic G-bundle* $\mathcal{U} \to C \times X$ *such that for any* $x \in X$ *the G-bundle* $\mathcal{U}_x := \mathcal{U}_{|C\times x}$ *is isomorphic with* $\varphi(x)$ *(where* φ *is the map of §1.4). Moreover, the bundle* $\mathcal{U}_{|C^*\times X}$ *comes equipped with a trivialization* $\alpha : \epsilon \tilde{\to} \mathcal{U}_{|C^*\times X}$, *where* ϵ *is the trivial G-bundle on* $C^* \times X$.

(b) Let $\mathcal{E} \to C \times Y$ *be an algebraic family of G-bundles (parametrized by an algebraic variety Y), such that* $\mathcal{E}$ *is trivial over* $C^* \times Y$ *and also over (Spec* $\hat{O}_p) \times Y$. *Then, if we choose a trivialization* $\beta : \epsilon' \tilde{\to} \mathcal{E}_{|C^*\times Y}$, *we get a Schubert variety* $X_{\mathfrak{w}}$ *and a unique morphism* $f : Y \to X_{\mathfrak{w}}$ *together with a G-bundle morphism* $\hat{f} : \mathcal{E} \to \mathcal{U}_{|C\times X_{\mathfrak{w}}}$ *inducing the map* $I\times f$ *at the base such that* $\hat{f} \circ \beta = \alpha \circ \theta$, *where* ϵ' *is the trivial bundle on* $C^* \times Y$ *and* θ *is the canonical G-bundle morphism* $\epsilon' \to \epsilon$ *inducing the map* $I\times f$ *at the base.*

Proof. Let R be a $\mathbb{C}$-algebra and let $Y :=$ Spec R be the corresponding scheme. Suppose $\mathcal{E} \to C \times Y$ is a G-bundle with trivializations β of $\mathcal{E}$ over $C^* \times Y$ and τ of $\mathcal{E}$ over (Spec $\hat{O}_p) \times Y$. Note that the fiber product (Spec $\hat{O}_p \times Y) \times_{C\times Y} (C^* \times Y)$ is canonically isomorphic with (Spec $\hat{k}_p) \times Y$ (cf. §1.4). Therefore the trivializations β and τ give rise to an element $\beta\tau^{-1} \in G(\hat{k}_p \otimes R)$. Conversely, given an element $g \in G(\hat{k}_p \otimes R)$, we can construct the family $\mathcal{E} \to C \times Y$ by taking the trivial bundles on $C^* \times Y$ and (Spec $\hat{O}_p) \times Y$ and glueing them via the element g (cf. §1.4). Moreover, if g_1 and g_2 are two elements of $G(\hat{k}_p \otimes R)$ such that $g_2 = g_1 h$ with $h \in G(\hat{O}_p \otimes R)$, then h induces a canonical isomorphism of the bundles corresponding to g_1 and g_2. All these assertions are easily verified.

Choose a local parameter t around $p \in C$. Let $\mathrm{ev}_\infty : G(\mathbb{C}[t^{-1}]) \to G$ be the group homomorphism induced from the algebra homomorphism $\mathbb{C}[t^{-1}] \to \mathbb{C}$ taking $t^{-1} \mapsto 0$, and let $N^- := \ker(\mathrm{ev}_\infty)$. Then the image U^- of N^- in X under the map $i : N^- \to X$, taking $g \mapsto g\mathcal{P}$, is an open subset of X. To construct a family of $G-$bundles on X, we first construct a family on the open sets $wU^- \subset X$, for $w \in \mathrm{Mor}_1(\mathbb{C}^*, T)$ as follows (cf. proof of Lemma C.10 for the notation $\mathrm{Mor}_1(\mathbb{C}^*, T)$):

From the discussion in the first paragraph, it suffices to construct an element $\theta_w \in G(\hat{k}_p \otimes \mathbb{C}[wU^-])$ such that, for every $wx \in wU^-$, the element θ_w evaluated at wx (i.e. the image of θ_w under the evaluation map $G(\hat{k}_p \otimes \mathbb{C}[wU^-]) \to G(\hat{k}_p)$ at wx) satisfies $\theta_w(wx) = wi^{-1}(x)$ mod $\mathcal{P}$. But, by definition, $N^- \subset G(\mathbb{C}[t^{-1}])$ and hence we get a tautological map $\theta : (\mathbb{P}^1(\mathbb{C})\backslash 0) \times N^- \to G$. It is easy to see that θ is a morphism under the ind-variety structure on N^-. (Observe that U^- being an open subset of X_{lat} has an ind-variety structure and hence N^- acquires an ind-variety structure via the bijection i.) Think of $\mathbb{C}^* = \mathbb{P}^1(\mathbb{C})\backslash\{0,\infty\}$ and define $\overline{\theta}_w : \mathbb{P}^1(\mathbb{C})\backslash\{0,\infty\} \times wU^- \to G$ by $\overline{\theta}_w(z, wi(g)) = w(z)\theta(z,g)$, for $z \in \mathbb{P}^1(\mathbb{C})\backslash\{0,\infty\}$ and $g \in N^-$. The morphism $\overline{\theta}_w$ of course gives rise to an element $\theta_w \in G(\hat{k}_p \otimes \mathbb{C}[wU^-])$, and hence a G-bundle on $C \times wU^-$.

To prove that the bundles on $C \times wU^-$ got from the elements θ_w patch together to give a bundle on $C \times X$, it suffices to show that the map

$$\overline{\theta}_v^{-1}\overline{\theta}_w : \mathbb{P}^1(\mathbb{C})\backslash\{0,\infty\} \times (wU^- \cap vU^-) \to G$$

extends to a morphism (again denoted by) $\overline{\theta}_v^{-1}\overline{\theta}_w : \mathbb{P}^1(\mathbb{C})\backslash\{\infty\} \times (wU^- \cap vU^-) \to G$: But for any fixed $x \in wU^- \cap vU^-$, the map $\overline{\theta}_v^{-1}\overline{\theta}_w : \mathbb{P}^1(\mathbb{C})\backslash\{0,\infty\} \times x \to G$ in