Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1128

Johannes Elschner

Singular Ordinary Differential Operators and Pseudodifferential Equations

Springer-Verlag
Berlin Heidelberg New York Tokyo

Lecture Notes in Mathematics

For information about Vols. 1–1034, please contact your bookseller of Springer-Verlag.

Vol. 1035: The Mathematics and Physics of Disordered Media. Proceedings, 1983. Edited by B.D. Hughes and B.W. Ninham. VII, 432 pages. 1983.

Vol. 1036: Combinatorial Mathematics X. Proceedings, 1982. Edited by L.R.A. Casse. XI, 419 pages. 1983.

Vol. 1037: Non-linear Partial Differential Operators and Quantization Procedures. Proceedings, 1981. Edited by S.I. Andersson and H.-D. Doebner. VII, 334 pages. 1983.

Vol. 1038: F. Borceux, G. Van den Bossche, Algebra in a Localic Topos with Applications to Ring Theory. IX, 240 pages. 1983.

Vol. 1039: Analytic Functions, Błażejewko 1982. Proceedings. Edited by J. Ławrynowicz. X, 494 pages. 1983

Vol. 1040: A. Good, Local Analysis of Selberg's Trace Formula. III, 128 pages. 1983.

Vol. 1041: Lie Group Representations II. Proceedings 1982–1983. Edited by R. Herb, S. Kudla, R. Lipsman and J. Rosenberg. IX, 340 pages. 1984.

Vol. 1042: A. Gut, K.D. Schmidt, Amarts and Set Function Processes. III, 258 pages. 1983.

Vol. 1043: Linear and Complex Analysis Problem Book. Edited by V.P. Havin, S.V. Hruščëv and N.K. Nikol'skii. XVIII, 721 pages. 1984.

Vol. 1044: E. Gekeler, Discretization Methods for Stable Initial Value Problems. VIII, 201 pages. 1984.

Vol. 1045: Differential Geometry. Proceedings, 1982. Edited by A.M. Naveira. VIII, 194 pages. 1984.

Vol. 1046: Algebraic K–Theory, Number Theory, Geometry and Analysis. Proceedings, 1982. Edited by A. Bak. IX, 464 pages. 1984.

Vol. 1047: Fluid Dynamics. Seminar, 1982. Edited by H. Beirão da Veiga. VII, 193 pages. 1984.

Vol. 1048: Kinetic Theories and the Boltzmann 1981. Edited by C. Cercignani. VII, 248 pages.

Vol. 1049: B. Iochum, Cônes autopolaires et a VI, 247 pages. 1984.

Vol. 1050: A. Prestel, P. Roquette, Formally p-adic 1984.

Vol. 1051: Algebraic Topology, Aarhus 1982. Proceedings. Edited by I. Madsen and B. Oliver. X, 665 pages. 1984.

Vol. 1052: Number Theory. Seminar, 1982. Edited by D.V. Chudnovsky, G.V. Chudnovsky, H. Cohn and M.B. Nathanson. V, 309 pages. 1984.

Vol. 1053: P. Hilton, Nilpotente Gruppen und nilpotente Räume. V, 221 pages. 1984.

Vol. 1054: V. Thomée, Galerkin Finite Element Methods for Parabolic Problems. VII, 237 pages. 1984.

Vol. 1055: Quantum Probability and Applications to the Quantum Theory of Irreversible Processes. Proceedings, 1982. Edited by L. Accardi, A. Frigerio and V. Gorini. VI, 411 pages. 1984.

Vol. 1056: Algebraic Geometry. Bucharest 1982. Proceedings, 1982. Edited by L. Bădescu and D. Popescu. VII, 380 pages. 1984.

Vol. 1057: Bifurcation Theory and Applications. Seminar, 1983. Edited by L. Salvadori. VII, 233 pages. 1984.

Vol. 1058: B. Aulbach, Continuous and Discrete Dynamics near Manifolds of Equilibria. IX, 142 pages. 1984.

Vol. 1059: Séminaire de Probabilités XVIII, 1982/83. Proceedings. Edité par J. Azéma et M. Yor. IV, 518 pages. 1984.

Vol. 1060: Topology. Proceedings, 1982. Edited by L.D. Faddeev and A.A. Mal'cev. VI, 389 pages. 1984.

Vol. 1061: Séminaire de Théorie du Potentiel. Paris, No. 7. Proceedings. Directeurs: M. Brelot, G. Choquet et J. Deny. Rédacteurs: F. Hirsch et G. Mokobodzki. IV, 281 pages. 1984.

Vol. 1062: J. Jost, Harmonic Maps Between Surfaces. X, 133 pages. 1984.

Vol. 1063: Orienting Polymers. Proceedings, 1983. Edited by J.L. Ericksen. VII, 166 pages. 1984.

Vol. 1064: Probability Measures on Groups VII. Proceedings, 1983. Edited by H. Heyer. X, 588 pages. 1984.

Vol. 1065: A. Cuyt, Padé Approximants for Operators: Theory and Applications. IX, 138 pages. 1984.

Vol. 1066: Numerical Analysis. Proceedings, 1983. Edited by D.F. Griffiths. XI, 275 pages. 1984.

Vol. 1067: Yasuo Okuyama, Absolute Summability of Fourier Series and Orthogonal Series. VI, 118 pages. 1984.

Vol. 1068: Number Theory, Noordwijkerhout 1983. Proceedings. Edited by H. Jager. V, 296 pages. 1984.

Vol. 1069: M. Kreck, Bordism of Diffeomorphisms and Related Topics. III, 144 pages. 1984.

Vol. 1070: Interpolation Spaces and Allied Topics in Analysis. Proceedings, 1983. Edited by M. Cwikel and J. Peetre. III, 239 pages. 1984.

Vol. 1071: Padé Approximation and its Applications, Bad Honnef 1983. Prodeedings. Edited by H. Werner and H.J. Bünger. VI, 264 pages. 1984.

Vol. 1072: F. Rothe, Global Solutions of Reaction-Diffusion Systems. V, 216 pages. 1984.

Vol. 1073: Graph Theory, Singapore 1983. Proceedings. Edited by K.M. Koh and H.P. Yap. XIII, 335 pages. 1984.

Vol. 1074: E.W. Stredulinsky, Weighted Inequalities and Degenerate Elliptic Partial Differential Equations. III, 143 pages. 1984.

Vol. 1075: H. Majima, Asymptotic Analysis for Integrable Connections with Irregular Singular Points. IX, 159 pages. 1984.

...Dimensional Systems. Proceedings, 1983. Edited
Schappacher. VII, 278 pages. 1984.

...epresentations III. Proceedings, 1982–1983.
Lipsman, J. Rosenberg. XI, 454

...an der Steen, Integration Theory.

Vol. 10... Ruppert. Compact Semitopological Semigroups: An Intrinsic Theory. V, 260 pages. 1984

Vol. 1080: Probability Theory on Vector Spaces III. Proceedings, 1983. Edited by D. Szynal and A. Weron. V, 373 pages. 1984.

Vol. 1081: D. Benson, Modular Representation Theory: New Trends and Methods. XI, 231 pages. 1984.

Vol. 1082: C.-G. Schmidt, Arithmetik Abelscher Varietäten mit komplexer Multiplikation. X, 96 Seiten. 1984.

Vol. 1083: D. Bump, Automorphic Forms on GL (3,IR). XI, 184 pages. 1984.

Vol. 1084: D. Kletzing, Structure and Representations of Q-Groups. VI, 290 pages. 1984.

Vol. 1085: G.K. Immink, Asymptotics of Analytic Difference Equations. V, 134 pages. 1984.

Vol. 1086: Sensitivity of Functionals with Applications to Engineering Sciences. Proceedings, 1983. Edited by V. Komkov. V, 130 pages. 1984

Vol. 1087: W. Narkiewicz, Uniform Distribution of Sequences of Integers in Residue Classes. VIII, 125 pages. 1984.

Vol. 1088: A.V. Kakosyan, L.B. Klebanov, J.A. Melamed, Characterization of Distributions by the Method of Intensively Monotone Operators. X, 175 pages. 1984.

Vol. 1089: Measure Theory, Oberwolfach 1983. Proceedings. Edited by D. Kölzow and D. Maharam-Stone. XIII, 327 pages. 1984.

Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1128

Johannes Elschner

Singular Ordinary Differential Operators and Pseudodifferential Equations

Springer-Verlag
Berlin Heidelberg New York Tokyo

Author

Johannes Elschner
Akademie der Wissenschaften der DDR, Institut für Mathematik
Mohrenstr. 39, 1086 Berlin, German Democratic Republic

This book is also published by Akademie-Verlag Berlin as volume 22 of the series "Mathematische Forschung".

AMS Subject Classification (1980): 35J70, 41A15, 45E05, 45L10, 47E05, 47G05

ISBN 3-540-15194-X Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 0-387-15194-X Springer-Verlag New York Heidelberg Berlin Tokyo

© 1985 Akademie-Verlag DDR-1086 Berlin
Printed in the German Democratic Republic

Printing: VEB Kongreß- und Werbedruck, DDR-9273 Oberlungwitz
Binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2146/3140-543210

此为试读,需要完整PDF请访问: www.ertongbook.com

To

Doris and Ulrike

INTRODUCTION

Various problems in physics and engineering lead to a linear ordinary differential equation whose coefficient of the highest derivative vanishes at certain points. Such an equation is called degenerate or singular, and a zero of the leading coefficient is said to be a singular point or singularity of the corresponding differential operator. In Chapters 1 and 2 of these notes we consider linear ordinary differential operators with a singular point at the origin:

$$A = x^q D^l + \sum_{0 \le i < l} a_i(x) D^i \ , \ D = d/dx \ , \ q \in \mathbb{N} \ . \tag{1}$$

In case of the homogeneous equation $Ay = 0$ with analytic coefficients a_i, the investigation of singular differential equations has a long history. In particular, starting with the works of Fuchs and Poincaré in the second half of the last century, the asymptotic behaviour as $x \to 0$ of solutions to such equations has been studied by many authors; see e.g. Sternberg [1], Wasow [1], Ince [1]. In contrast to that, general results on the solvability of the degenerate inhomogeneous equation $Ay = f$ have been obtained only during the last fifteen years, using the methods of linear functional analysis.

In order to describe such a result and the material of these notes, it is necessary to recall some definitions. Let X and Y be linear topological spaces, and $A : X \to Y$ a linear operator with the domain of definition $D(A) \subset X$. If A is a continuous map of X into Y and $D(A) = X$, we shall write $A \in L(X,Y)$ and $A \in L(X)$ when $X = Y$. A is called normally solvable if its range $\operatorname{im} A = A(D(A))$ is closed in Y. The dimension $\dim \ker A$ of the kernel $\ker A = \{x \in D(A) : Ax = 0\}$ will be called the kernel index or nullity of A, and the deficiency $\dim Y/\operatorname{im} A$ of $\operatorname{im} A$ in Y will be called the deficiency index of A. If A is normally solvable and its kernel and deficiency indices are both finite, we say that A is a Fredholm operator, and its index is defined by $\operatorname{ind} A = \dim \ker A - \dim Y/\operatorname{im} A$. Furthermore, A is called Φ_+- (resp. Φ_--)operator if it is normally solvable and $\dim \ker A < \infty$ (resp. $\dim Y/\operatorname{im} A < \infty$). For Fredholm and $\Phi_\pm$-operators and their basic properties, we refer to the

exposition in Goldberg [1], Prößdorf [1], Przeworska-Rolewicz and Rolewicz [1].

Malgrange [1], Komatsu [1] and Korobejnik [1] independently of each other found a general index formula for the operator (1) in the space $\mathcal{H}(\Omega)$ of analytic functions in a domain $\Omega \subset \mathbb{C}$, $0 \in \Omega$, which we state here in the special case of a simply connected domain:

If $a_i \in \mathcal{H}(\Omega)$ $(i=0,\dots,l-1)$, then $A \in L(\mathcal{H}(\Omega))$ is a Fredholm operator with index $l-q$.

Thus we observe that the index of A coincides with the index of the principal part $x^q D^l$ in $\mathcal{H}(\Omega)$. This result is not true, in general, if (1) acts in spaces of differentiable functions on an interval $[a,b] \subset \mathbb{R}$, $0 \in [a,b]$. Since the end of the sixties there appeared a lot of papers concerning the index and the solvability properties of the operator (1) in such spaces. The aim of Chapters 1 and 2 of this work is to fit most of those results into a general framework and thus to give a review of the current state of this field. Chapter 1 deals with the special case of a Fuchsian differential operator and also serves as an illustration of typical methods and results in the theory of singular ordinary differential equations. In Chapter 2 we give a general index formula for the differential operator (1) in the space of infinitely differentiable functions and in weighted L_p spaces on an interval. Furthermore, we study kernel, range, normal solvability in weighted Sobolev spaces and hypoellipticity of the degenerate operator (1) as well as its index in the space of distributions. A preliminary version of Chapters 1 and 2 appeared in Elschner and Silbermann [2].

In Chapter 3 these results will partly be generalized to differential operators with a finite number of singular points on compact or infinite intervals. In case of Fuchsian differential operators on a finite interval, the index formula is applied to study essential selfadjointness and spectrum of boundary value problems for those operators.

We remark that there is an increased interest in the development of the theory of singular ordinary differential operators in connection with the very active fields of solvability theory for degenerate partial dif-

ferential equations (see e.g. Bolley, Camus and Helffer [1], [2], Baouendi and Goulaouic [1], Baouendi, Goulaouic and Lipkin [1], Helffer and Rodino [1], Višik and Grušin [1], Elschner and Lorenz [3], Lorenz [1]) and numerical analysis for degenerate ordinary differential equations (cf. e.g. de Hoog and Weiss [1], [2], Natterer [1], Elschner and Silbermann [1]).

Chapter 4 illustrates the application of singular ordinary differential operators in the theory of partial differential equations. Relying on certain results in Chapters 1 and 2, we study local solvability as well as normal solvability and index in Sobolev spaces for some examples of elliptic partial differential operators degenerating at one point. Many problems in this field are still open.

Various boundary value problems in mathematical physics and complex function theory lead to singular integro-differential, or more generally, to pseudodifferential equations on a closed curve. A pseudodifferential operator is called classical if its symbol has an asymptotic expansion as a sum of symbols which are positive homogeneous in the covariable of decreasing orders (cf. Chapter 5); it is called non-elliptic or degenerate if the principal term of the symbol vanishes at certain points on the cosphere bundle of the curve.

A major part of Chapter 5 is devoted to the author's recent results on the index and the Fredholm property of degenerate classical pseudodifferential operators on a closed contour, though, for the lack of space, we have not covered all of the material in full generality. As an application of these results, theorems on the index and on existence and uniqueness of solutions of the degenerate oblique derivative problem in the plane are given. Furthermore, in Chapter 5 we have included an almost self-contained introduction to classical pseudodifferential operators on a closed curve. For the general theory of pseudodifferential equations, the reader is referred to Šubin [1], Taylor [1] and Treves [1]. An exposition of the theory of degenerate one-dimensional singular integral equations which have been studied somewhat earlier can be found in Prößdorf [1] and Michlin and Prößdorf [1].

Chapter 6 deals with the Galerkin method using periodic splines as test and trial functions for the approximate solution of pseudodifferential equations on a closed contour. It demonstrates the interplay between certain a priori estimates for pseudodifferential operators, namely the Gårding and Melin inequalities, and convergence results for Galerkin's method with splines for strongly elliptic and degenerate equations. For an introduction to the theory of splines and finite element methods, we refer to Aubin [1], de Boor [1] and Strang and Fix [1]. The reader should consult the introduction and the section "comments and references" in each chapter for more information on the contents of these notes and further references.

Except for Chapter 4, the material is rather self-contained. The reader is assumed to be familiar with linear functional analysis (see e.g. Goldberg [1], Prößdorf [1]). In Chapter 4 some previous knowledge of elliptic differential operators on manifolds is desirable (cf. Narasimhan [1], Agranovič and Višik [1]).

Throughout the book the following notation is used. For a domain $\Omega \subset \mathbb{R}^n$, let $C^\infty(\Omega)$ ($C_0^\infty(\Omega)$) be the set of infinitely differentiable functions (with compact support) in Ω, and $C^\infty(\bar{\Omega})$ the set of all infinitely differentiable functions in Ω which together with all derivatives continuously extend to the closure $\bar{\Omega}$ of Ω. For $\Omega = (a,b) \subset \mathbb{R}$, we simply write $C^\infty(\Omega) = C^\infty(a,b)$, $C^\infty(\bar{\Omega}) = C^\infty[a,b]$ etc. The support of a function u is denoted by supp u. In $C^\infty(\bar{\Omega})$ (resp. $C_0^\infty(\Omega)$) one can introduce the topology of a Fréchet (resp. locally convex) space; see Hörmander [2], Robertson and Robertson [1]. The bilinear form

$$\langle u,v \rangle = \int_\Omega uv \, dx$$

on $C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ extends to a duality between $C_0^\infty(\Omega)$ and the locally convex space $\mathcal{D}'(\Omega)$ of all distributions in Ω. For $A \in L(C_0^\infty(\Omega))$, the transpose ${}^tA \in L(\mathcal{D}'(\Omega))$ of A is defined by

$$\langle {}^tAu, v \rangle = \langle u, Av \rangle, \quad u \in \mathcal{D}', \quad v \in C_0^\infty.$$

Finally, if M is an n-dimensional infinitely differentiable manifold with or without boundary, let $C^\infty(M)$ be the set of all infinitely dif-

ferentiable functions on M; see Narasimhan [1]. Other notation is either standard or defined upon introduction.

Acknowledgements

My interest in degenerate differential and integral equations was stimulated by my teacher Prof. Dr. S. Prößdorf, and I thank him for helpful discussions and permanent attention in my scientific development. The results in Chapter 4 arose in cooperation with Dr. M. Lorenz, and I am grateful to him for going through the manuscript and making various suggestions and corrections. Further I want to express my gratitude to Dr. G. Schmidt who read parts of the manuscript. In particular, his remarks led to several improvements in Chapter 6. I would also like to thank Mrs. Ch. Huber for typing the present monograph. Finally, I want to acknowledge the Akademie-Verlag, especially Dr. R. Höppner, for effective cooperation and including this publication in the series "Mathematical Research".

Berlin, December 1983

J. Elschner

此为试读,需要完整PDF请访问: www.ertongbook.com

CONTENTS

1. Fuchsian differential operators with one singular point 13
1.1. Spaces 13
1.2. The Euler operator 16
1.3. The Fuchsian operator (I) 22
1.4. The Fuchsian operator (II) 25
1.5. Smoothness and Hölder continuity of solutions 28
1.6. Basis of solutions for the homogeneous equation 3o
1.7. Maximal Fuchsian operators 33
1.8. Comments and references 35

2. Linear ordinary differential operators with one singular point 37
2.1. Factorization of formal differential operators 37
2.2. On the computation of characteristic factors 44
2.3. Principal parts of differential operators 48
2.4. The index in weighted L_p spaces and C^∞ 54
2.5. Hypoellipticity and basis of solutions 59
2.6. A class of differential operators in weighted Sobolev spaces 62
2.7. Examples 69
2.8. Comments and references 7o

3. Linear ordinary differential operators with several singular points 72
3.1. Reduction to the case of one singularity 72
3.2. A class of selfadjoint differential operators with two singularities 74
3.3. Differential operators without singularities on an infinite interval 8o
3.4. Differential operators with one singularity on $(0,\infty)$ 85
3.5. Differential operators with a finite number of singularities on $(-\infty,\infty)$ 89
3.6. Comments and references 9o

4. Elliptic differential operators in $\mathbb{R}^n$ degenerating at one point 92
4.1. Preliminaries on function spaces and the Mellin transform 92
4.2. The operator $r^2\Delta + ar\partial/\partial r + b$ 1o1
4.3. A class of elliptic operators degenerating at the origin 1o8
4.4. A hypoelliptic differential operator which is not locally solvable 116
4.5. Comments and references 121

5. Degenerate pseudodifferential operators on a closed curve 123
5.1. Pseudodifferential operators on a closed curve 123
5.2. Gårding-Melin inequalities 134
5.3. The index for a class of degenerate pseudodifferential operators 137
5.4. Smoothness of solutions and special cases 146
5.5. On the degenerate oblique derivative problem in the plane 151
5.6. A class of pseudodifferential operators degenerating at one point 156
5.7. Comments and references 161

6. A finite element method for pseudodifferential equations on a closed curve 164
6.1. Spline approximation in periodic Sobolev spaces 164
6.2. Spline approximation in weighted Sobolev spaces 17o
6.3. The Galerkin method with splines for strongly elliptic equations 173
6.4. The Galerkin method with splines for degenerate equations 177
6.5. Comments and references 185
7. Appendix. Suboptimal convergence of the Galerkin method with splines for elliptic pseudodifferential equations 187
References 191
List of basic symbols 197
Subject index 199

1. FUCHSIAN DIFFERENTIAL OPERATORS WITH ONE SINGULAR POINT

In this chapter we consider the Fuchsian differential operator

$$(Ay)(x) = [x^l D^l + \sum_{0 \le i < l} a_i(x) x^i D^i] y(x), \quad Dy = y' = dy/dx , \qquad (1.0.1)$$

of order l which has a singular point at the origin. We have chosen this simple class of operators in order to introduce the methods which are used in the study of more general degenerate ordinary differential equations in Chap. 2. Furthermore, Fuchsian differential equations occur in several applications in mathematical physics and mechanics, which justifies a more detailed investigation of those operators.

Before studying the operator (1.0.1) in certain spaces of differentiable functions, we consider the corresponding Euler differential operator

$$A_0 = x^l D^l + \sum_{0 \le i < l} a_i(0) x^i D^i . \qquad (1.0.2)$$

It turns out that A and A_0 are simultaneously Fredholm operators in the corresponding spaces and their indices coincide since A is a small perturbation of A_0 in a certain sense. Thus A_0 can be considered as the principal part of A.

1.1. Spaces

In this section we define function spaces in which index and solvability properties of the operator (1.0.1) shall be investigated, and collect some of their properties.

1.1.1. Let $b > 0$ and $L_p(0,b)$, $1 \le p \le \infty$, be the space of all complex-valued measurable functions on $(0,b)$ for which the norm

$$\| y \|_{L_p(0,b)} = \left(\int_0^b |y(x)|^p dx \right)^{1/p} , \quad 1 \le p < \infty ,$$

$$\| y \|_{L_\infty(0,b)} = \operatorname*{ess\,sup}_{0 \le x \le b} |y(x)|$$

is finite. For $k \in \mathbb{N}$, $W_p^k(0,b)$ will denote the Sobolev space of all functions y for which $y^{(k-1)} = D^{k-1} y$ exists and is absolutely continuous on $[0,b]$ and the norm given by

$$\|y\|_{W_p^k(0,b)} = \sum_{0\le i\le k} \|y^{(i)}\|_{L_p(0,b)}$$

is finite. We set $L_p = W_p^0$. Let $C^k[0,b]$, $k\in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, be the space of k times continuously differentiable functions on $[0,b]$ with norm

$$\|y\|_{C^k[0,b]} = \sum_{0\le i\le k} \max_{0\le x\le b} |y^{(i)}(x)| .$$

Furthermore, we introduce the spaces with weights

$$L_p^\varrho(0,b) = W_p^{0,\varrho}(0,b) = \{ x^\varrho y : y\in L_p(0,b)\} , \quad \varrho\in\mathbb{R},$$

$$C^{0,\varrho}[0,b] = \{ y\in C^0[0,b] : x^{-\varrho}[y(x)-y(0)]\in C^0[0,b]\} , \quad \varrho\ge 0 ,$$

and for $k\in\mathbb{N}$, $\varrho\ge 0$,

$$W_p^{k,\varrho}(0,b) = \{y\in W_p^{k-1}(0,b) : y^{(k)}\in L_p^\varrho(0,b)\} ,$$

$$C^{k,\varrho}[0,b] = \{ y\in C^{k-1}[0,b] : y^{(k)}\in C^{0,\varrho}[0,b]\} .$$

$W_p^{k,\varrho}$ and $C^{k,\varrho}$ are Banach spaces with norms defined by

$$\|y\|_{W_p^{k,\varrho}(0,b)} = \|y\|_{W_p^{k-1}(0,b)} + \|x^{-\varrho}y^{(k)}\|_{L_p(0,b)} ,$$

$$\|y\|_{C^{k,\varrho}[0,b]} = \|y\|_{C^{k-1}[0,b]} + \|x^{-\varrho}[y^{(k)}(x)-y^{(k)}(0)]\|_{C^0[0,b]} + |y^{(k)}(0)| ,$$

where the first term on the right-hand sides is omitted for k=0.

Finally, for $k\in\mathbb{N}_0$ and $\lambda\in(0,1]$, let $H^{k,\lambda}[0,b]$ be the space of all functions $y\in C^k[0,b]$ for which $y^{(k)}$ satisfies a Hölder condition with exponent λ on $[0,b]$, equipped with the norm

$$\|y\|_{H^{k,\lambda}[0,b]} = \|y\|_{C^k[0,b]} + \sup_{0\le x<t\le b} |y(x)-y(t)|\,|x-t|^{-\lambda} .$$

1.1.2. For $\varrho\ge 0$, we introduce the closed subspaces

$$\overset{\circ}{W}{}_p^{k,\varrho}(0,b) = \{ y\in W_p^{k,\varrho}(0,b) : y^{(i)}(0) = 0,\ i=0,\dots,k-1\}, \quad k\in\mathbb{N},$$

$$\overset{\circ}{C}{}^{k,\varrho}[0,b] = \{ y\in C^{k,\varrho}[0,b] : y^{(i)}(0) = 0,\ i=0,\dots,k\} , \quad k\in\mathbb{N}_0,$$

of $W_p^{k,\varrho}$ and $C^{k,\varrho}$, respectively. It is easy to check that

$$\|y\|_{\mathring{W}_p^{k,\varrho}} = \|x^{-\varrho}y^{(k)}\|_{L_p}, \quad \|y\|_{\mathring{C}^{k,\varrho}} = \|x^{-\varrho}y^{(k)}\|_{C^0} \tag{1.1.1}$$

are equivalent norms in $\mathring{W}_p^{k,\varrho}$ and $\mathring{C}^{k,\varrho}$, respectively. We set $L_p^{\varrho} = W_p^{0,\varrho} = \mathring{W}_p^{0,\varrho}$ ($\varrho \in \mathbb{R}$). Finally, let

$$\mathring{H}^{k,\lambda}[0,b] = \{y \in H^{k,\lambda}[0,b] : y^{(i)}(0) = 0,\ i=0,\dots,k\}.$$

Then

$$\|y\|_{\mathring{H}^{k,\lambda}[0,b]} = \sup_{0<x\le t<b} |y^{(k)}(x)-y^{(k)}(t)|\,|x-t|^{-\lambda} \tag{1.1.2}$$

is an equivalent norm in the subspace $\mathring{H}^{k,\lambda}$ of $H^{k,\lambda}$.

1.1.3. For $l \in \mathbb{N}$, let

$$W_{p,l}^{k,\varrho}(0,b) = \{y \in W_p^{k,\varrho}(0,b) : x^iD^iy \in W_p^{k,\varrho}(0,b),\ i=1,\dots,l\},$$

where the terms x^iD^iy are defined in the sense of distributions. $W_{p,l}^{k,\varrho}$ is a Banach space with the canonical norm

$$\|y\|_{W_{p,l}^{k,\varrho}(0,b)} = \sum_{0\le i\le l} \|x^iD^iy\|_{W_p^{k,\varrho}(0,b)}.$$

Analogously

$$C_l^{k,\varrho}[0,b] = \{y \in C^{k,\varrho}[0,b] : x^iD^iy \in C^{k,\varrho}[0,b],\ i=1,\dots,l\}$$

is a Banach space endowed with the canonical norm. Since x^iD^i is a linear combination of the terms $(xD)^j$ ($j \le i$),

$$|y|_{W_{p,l}^{k,\varrho}} = \sum_{0\le i\le l} \|(xD)^iy\|_{W_p^{k,\varrho}}, \quad |y|_{C_l^{k,\varrho}} = \sum_{0\le i\le l} \|(xD)^iy\|_{C^{k,\varrho}} \tag{1.1.3}$$

are equivalent norms in $W_{p,l}^{k,\varrho}$ and $C_l^{k,\varrho}$, respectively. We shall also write $L_{p,l}^{\varrho}$ instead of $W_{p,l}^{0,\varrho}$.

Lemma 1.1.1. (i) For any $y \in C_l^{k,\varrho}[0,b]$,

$$D^j(x^iD^iy)|_{x=0} = j(j-1)\cdots(j-i+1)(D^jy)(0), \quad j=0,\dots,k,\ i=1,\dots,l. \tag{1.1.4}$$

(ii) If $y \in W_{p,l}^{k,\varrho}(0,b)$, $k \in \mathbb{N}$, then (1.1.4) holds for $j=0,\dots,k-1$.

Proof. (i) Let $j=0$ and $i=1$. Since $xDy = Dxy-y$ on $(0,b]$ and

$$Dxy|_{x=0} = \lim_{x\to 0} xy/x = y(0),$$

we obtain $xDy|_{x=0} = 0$. Using the identities

此为试读,需要完整PDF请访问: www.ertongbook.com