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Global Bifurcations and Chaos

Analytical Methods

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Stephen Wiggins

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Analytical Methods

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PREFACE

The study of chaotic phenomena in deterministic nonlinear dynamical systems has attracted much attention over the last fifteen years. For the applied scientist, this study poses three fundamental questions. First, and most simply, what is meant by the term "chaos"? Second, by what mechanisms does chaos occur, and third, how can one predict when it will occur in a specific dynamical system? This book begins the development of a program that will answer these questions.

I have attempted to make the book as self-contained as possible, and thus have included some introductory material in Chapter One. The reader will find much new material in the remaining chapters. In particular, in Chapter Two, the techniques of Conley and Moser (Moser [1973]) and Afraimovich, Bykov, and Silnikov [1983] for proving that an invertible map has a hyperbolic, chaotic invariant Cantor set are generalized to arbitrary (finite) dimensions and to subshifts of finite type. Similar techniques are developed for the nonhyperbolic case. These nonhyperbolic techniques allow one to demonstrate the existence of a chaotic invariant set having the structure of the Cartesian product of a Cantor set with a surface or a "Cantor set of surfaces". In Chapter Three the nonhyperbolic techniques are applied to the study of the orbit structure near orbits homoclinic to normally hyperbolic invariant tori.

In Chapter Four, I develop a class of global perturbation techniques that enable one to detect orbits homoclinic or heteroclinic to hyperbolic fixed points, hyperbolic periodic orbits, and normally hyperbolic invariant tori in a large class of systems. The methods developed in Chapter Four are similar in spirit to a technique originally developed by Melnikov [1963] for periodically forced, two-dimensional systems;

however, they are much more general in that they are applicable to arbitrary (but finite) dimensional systems and allow for slowly varying parameters and quasiperiodic excitation. This general theory will hopefully be of interest to the applied scientist, since it allows one to give a criterion for chaotic dynamics in terms of the system parameters. Moreover, the methods apply in arbitrary dimensions, where much work remains to be done in chaos and nonlinear dynamics.

In this book I do not deal with the question of the existence of strange attractors. Indeed, this remains a major outstanding problem in the subject. However, this book does provide useful techniques for studying strange attractors, in that the first step in proving that a system possesses a chaotic attracting set is to prove that it possesses chaotic dynamics and then to show that the dynamics are contained in an attracting set that has no stable "regular" motions. One cannot deny that chaotic Cantor sets can radically influence the dynamics of a system; however, the extent and nature of this influence needs to be studied. This will require the development of new ideas and techniques.

Over the past two years many people have offered much encouragement and help in this project, and I take great pleasure in thanking them now.

Phil Holmes and Jerry Marsden gave me the initial encouragement to get started and criticized several early versions of the manuscript.

Steve Schecter provided extremely detailed criticisms of early versions of the manuscript which prevented many errors.

Steve Shaw read and commented on all of the manuscript.

Pat Sethna listened patiently to my explanations of various parts of the book and helped me considerably in clarifying my thoughts and presentation style.

John Allen and Roger Samelson called my attention to a crucial error in some earlier work.

Darryl Holm, Daniel David, and Mike Tratnik listened to several lengthy explanations of material in Chapters Three and Four and pointed out several errors in the manuscript.

Much of the material in Chapters Three and Four was first tried out in graduate applied math courses at Caltech. I am grateful to the students in those courses for enduring many obscure lectures and offering useful suggestions.

During the past two years Donna Gabai and Jan Patterson worked tirelessly on the layout and typing of this manuscript. They unselfishly gave of their time (often evenings and weekends) so that various deadlines could be met. Their skill and help made the completion of this book immensely easier.

I would also like to acknowledge the artists who drew the figures for this book and pleasantly tolerated my many requests for revisions. The figures for Chapter One were done by Betty Wood, and those for Chapter Four by Cecilia Lin. Peggy Firth, Pat Marble, and Bob Turring of the Caltech Graphic Arts Facilities and Joe Pierro, Haydee Pierro, Melissa Loftis, Gary Hatt, Marcos Prado, Bill Contado, Abe Won, and Stacy Quinet of Imperial Drafting Inc. drew the figures for Chapters Two and Three.

Finally, Meredith Allen gave indispensable advice and editorial assistance throughout this project.

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CHAPTER 1
Introduction: Background for Ordinary Differential Equations and Dynamical Systems

The purpose of this first chapter is to review and develop the necessary concepts from the theory of ordinary differential equations and dynamical systems which we will need for the remainder of the book. We will begin with some results from classical ordinary differential equations theory such as existence and uniqueness of solutions, dependence of solutions on initial conditions and parameters, and various concepts of stability. We will then discuss more modern ideas such as genericity, structural stability, bifurcations, and Poincaré maps. Standard references for the theory of ordinary differential equations are Coddington and Levinson [1955], Hale [1980], and Hartman [1964]. We will take a more global, geometric point of view of the theory; some references which share this viewpoint are Arnold [1973], Guckenheimer and Holmes [1983], Hirsch and Smale [1974], and Palis and deMelo [1982].

1.1. The Structure of Solutions of Ordinary Differential Equations

In this book we will regard an ordinary differential equation as a system of equations having the following form

$$\dot{x} = f(x,t) , \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}^1 \tag{1.1.1}$$

where $f : U \to \mathbb{R}^n$ with U an open set in $\mathbb{R}^n \times \mathbb{R}^1$ and $\dot{x} \equiv dx/dt$. The space of dependent variables is often referred to as the *phase* or *state space* of the system (1.1.1). By a solution of (1.1.1) we will mean a map

$$\phi : I \to \mathbb{R}^n \tag{1.1.2}$$

where I is some interval in $\mathbb{R}$ such that

$$\dot{\phi}(t) = f(\phi(t), t) \ . \tag{1.1.3}$$

Thus, geometrically (1.1.1) can be viewed as defining a vector at every point in U, and a solution of (1.1.1) is a curve in $\mathbb{R}^n$ whose tangent or velocity vector at each point is given by $f(x,t)$ evaluated at the specific point. For this reason (1.1.1) is often referred to as a *vector field*.

Now, the existence of solutions of (1.1.1) is certainly not obvious and evidently must rely in some way on the properties of f, so now we want to give some classical results concerning existence of solutions of (1.1.1) and their properties.

1.1a. Existence and Uniqueness of Solutions

Suppose that f is C^r in U (note: by C^r, $r \geq 1$, we mean that f has r derivatives which are continuous at each point of U; C^0 means that f is continuous at each point of U) and for some $\epsilon_1, \epsilon_2 > 0$ let $I_1 = \{t \in \mathbb{R} \mid t_0 - \epsilon_1 < t < t_0 + \epsilon_1\}$ and $I_2 = \{t \in \mathbb{R} \mid t_0 - \epsilon_2 < t < t_0 + \epsilon_2\}$; then we have the following theorem.

Theorem 1.1.1. *Let (x_0, t_0) be a point in U. Then for ϵ_1 sufficiently small there exists a solution of (1.1.1), $\phi_1 : I_1 \to \mathbb{R}^n$, satisfying $\phi_1(t_0) = x_0$. Moreover, if f is C^r in U, $r \geq 1$, and $\phi_2 : I_2 \to \mathbb{R}^n$ is also a solution of (1.1.1) satisfying $\phi_2(t_0) = x_0$, then $\phi_1(t) = \phi_2(t)$ for all $t \in I_3 = \{t \in \mathbb{R} \mid t_0 - \epsilon_3 < t < t_0 + \epsilon_3\}$ where $\epsilon_3 = \min\{\epsilon_1, \epsilon_2\}$.*

PROOF: See Arnold [1973] or Hale [1980]. □

We make the following remarks concerning Theorem 1.1.1.

1) For a solution of (1.1.1) to exist, only continuity of f is required; however, in this case the solution passing through a given point in U may not be unique (see Hale [1980] for an example). If f is at least C^1 in U, then there is only one solution passing through any given point of U (note: for uniqueness of solutions one actually only needs f to be Lipschitz in the x variable uniformly in t, see Hale [1980] for the proof). The degree of differentiability of the vector field will not be a major concern to us in this book since all of the examples we consider will be infinitely differentiable.

2) The differentiability of solutions with respect to t was not explicitly considered in the theorem, although evidently they must be at least C^r since f is C^r. This result will be stated shortly.
3) Notation: In denoting the solutions of (1.1.1) it may be useful to note the dependence on initial conditions explicitly. For ϕ, a solution of (1.1.1) passing through the point $x = x_0$ at $t = t_0$, the notation would be

$$\phi(t, t_0, x_0) \quad \text{with} \quad \phi(t_0, t_0, x_0) = x_0 . \tag{1.1.4}$$

In some cases, the initial time is always understood to be a specific value (often $t_0 = 0$); in this case, the explicit dependence on the initial time is omitted and the solution is written as

$$\phi(t, x_0) \quad \text{with} \quad \phi(t_0, x_0) = x_0 . \tag{1.1.5}$$

1.1b. Dependence on Initial Conditions and Parameters

In the computation of stability properties of solutions and in the construction of Poincaré maps (see Section 1.6) the differentiability of solutions with respect to initial conditions is very important.

Theorem 1.1.2. *If $f(x,t)$ is C^r in U, then the solution of 1.1.1, $\phi(t, t_0, x_0)$ $(x_0, t_0) \in U$, is a C^r function of t, t_0 and x_0.*

PROOF: See Arnold [1973] or Hale [1980]. □

Theorem 1.1.2 justifies the procedure of computing the Taylor series expansion of a solution of (1.1.1) about a given initial condition. This enables one to determine the nature of solutions near a particular solution. Often the linear term in such an expansion is sufficient for determining many of the local properties near a particular solution (e.g., stability). The following theorem gives an equation which the first derivative of the solution with respect to x_0 must obey.

Theorem 1.1.3. *Suppose $f(x,t)$ is C^r, $r \geq 1$, in U and let $\phi(t, t_0, x_0)$, $(x_0, t_0) \in U$, be a solution of (1.1.1). Then the $n \times n$ matrix $D_{x_0}\phi$ is the solution of the following linear ordinary differential equation*

$$\dot{Z} = D_x f\,(\phi(t), t)\; Z\,, \qquad Z(t_0) = \mathrm{id}\,, \tag{1.1.6}$$

where Z is an $n \times n$ matrix and **id** *denotes the $n \times n$ identity matrix.*

PROOF: See Arnold [1973], Hale [1980], or Irwin [1980]. □

Equation (1.1.6) is often referred to as the *first variational equation*. We remark that it is possible to find linear ordinary differential equations which the higher order derivatives of solutions with respect to the initial conditions must obey; however, we will not need these in this book.

Now suppose that equation (1.1.1) depends on parameters

$$\dot{x} = f(x,t;\epsilon)\,, \qquad (x,t,\epsilon) \in \mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^p \tag{1.1.7}$$

where $f : U \to \mathbb{R}^n$ with U an open set in $\mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^p$. We have the following theorem.

Theorem 1.1.4. *Suppose $f(x,t;\epsilon)$ is C^r in U. Then the solution of (1.1.7), $\phi(t,t_0,x_0,\epsilon)$ $(x_0,t_0,\epsilon) \in U$, is a C^r function of ϵ.*

PROOF: See Arnold [1973] or Hale [1980]. □

In many applications it is useful to seek Taylor series expansions in ϵ of solutions of (1.1.7) (e.g., in perturbation theory and bifurcation theory). Analogous to Theorem 1.1.3, the following theorem gives an ordinary differential equation which the first derivative of a solution of (1.1.7) with respect to ϵ must obey.

Theorem 1.1.5. *Suppose $f(x,t,\epsilon)$ is C^r, $r \geq 1$, in U and let $\phi(t,t_0,x_0,\epsilon)$, $(x_0,t_0,\epsilon) \in U$, be a solution of (1.1.7). Then the $n \times p$ matrix $D_\epsilon\phi$ satisfies the following linear ordinary differential equation*

$$\dot{Z} = D_x f\left(\phi(t),t;\epsilon\right) Z + D_\epsilon f\left(\phi(t),t;\epsilon\right)\,, \qquad z(t_0) = \mathbf{0}\,, \tag{1.1.8}$$

where Z is a $n \times p$ matrix and **0** *represents the $n \times p$ matrix of zeros.*

PROOF: See Hale [1980]. □

1.1c. Continuation of Solutions

Theorem 1.1.1 gave sufficient conditions for the existence of solutions of (1.1.1) but only on a sufficiently small time interval. We will now give a theorem which justifies the extension of this time interval, but first we need the following definition.

Definition 1.1.1. Let ϕ_1 be a solution of (1.1.1) defined on the interval I_1, and let ϕ_2 be a solution of (1.1.1) defined on the interval I_2. We say that ϕ_2 is a *continuation* of ϕ_1 if $I_1 \subset I_2$ and $\phi_1 = \phi_2$ on I_1. A solution is *noncontinuable* if no such continuation exists; in this case, I_1 is called the *maximal interval of existence* of ϕ_1.

We now state the following theorem concerning continuation of solutions.

Theorem 1.1.6. *Suppose $f(x,t)$ is C^r in U and $\phi(t,t_0,x_0)$, $(x_0,t_0) \in U$, is a solution of (1.1.1), then there is a continuation of ϕ to a maximal interval of existence. Furthermore, if (t_1,t_2) is a maximal interval of existence for ϕ, then $(\phi(t),t)$ tends to the boundary of U as $t \to t_1$ and $t \to t_2$.*

PROOF: See Hale [1980]. □

Terminology

At this point we want to introduce some common terminology that applies to solutions of ordinary differential equations. Recall that a solution of (1.1.1) is a map $\phi : I \to \mathbb{R}^n$ where I is some interval in $\mathbb{R}$. Geometrically, the image of I under ϕ is a curve in $\mathbb{R}^n$, and this geometrical picture gives rise to the following terminology.

1) A solution $\phi(t,t_0,x_0)$ of (1.1.1) may also be called the *trajectory, phase curve or motion* through the point x_0.
2) The graph of the solution $\phi(t,t_0,x_0)$, i.e.,

$$\left\{ (x,t) \in \mathbb{R}^n \times \mathbb{R}^1 \mid x = \phi(t,t_0,x_0),\ t \in I \right\}$$

   is called an *integral curve*.
3) Suppose we have a solution $\phi(t,t_0,x_0)$; then the set of points in $\mathbb{R}^n$ through which this solution passes as t varies through I is called the *orbit through* x_0, denoted $O(x_0)$ and written as follows.

$$O(x_0) = \{ x \in \mathbb{R}^n \mid x = \phi(t,t_0,x_0), \quad t \in I \} .$$

   We remark that it follows from this definition that, for any $T \in I$,

$$O\left(\phi(T,t_0,x_0)\right) = O(x_0) .$$

The following example should serve to illustrate the terminology.

EXAMPLE 1.1.1. Consider the following equation

$$\ddot{x} + x = 0\,. \tag{1.1.9}$$

This is just the equation for a simple harmonic oscillator having frequency one. Writing (1.1.9) as a system we obtain

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -x\,. \end{aligned} \tag{1.1.10}$$

Equation (1.1.10) has the form of equation (1.1.1) with phase space $\mathbb{R}^2$. The solution of (1.1.10) passing through the point (1,0) at $t = 0$ is given by $\phi(t) = (\cos t, -\sin t)$.

1) The *trajectory, phase curve or motion* through the point (1,0) is illustrated in Figure 1.1.1.

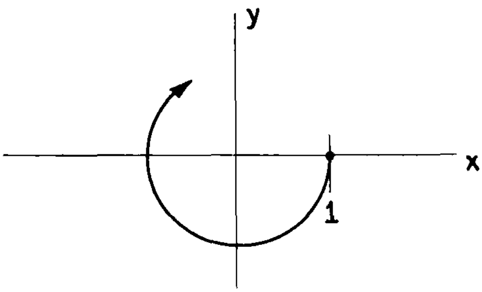

Figure 1.1.1. Trajectory through the Point (1,0).

2) The *integral curve* of the solution $\phi(t) = (\cos t, -\sin t)$ is illustrated in Figure 1.1.2.
3) The orbit through the point (1,0) is given by $\left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \right\}$ and is illustrated in Figure 1.1.3.

We remark that, although the solution through (1,0) passes through the same set of points in $\mathbb{R}^2$ as the orbit through (1,0), and thus both appear to be the same object when viewed as a locus of points in $\mathbb{R}^2$, we stress that they are indeed different objects. A solution must pass through a specific point at a specified time and an

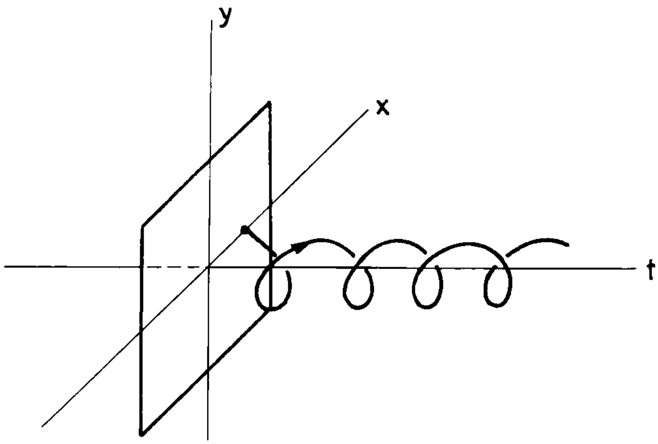

Figure 1.1.2. Integral Curve of $\phi(t) = (\cos t, -\sin t)$.

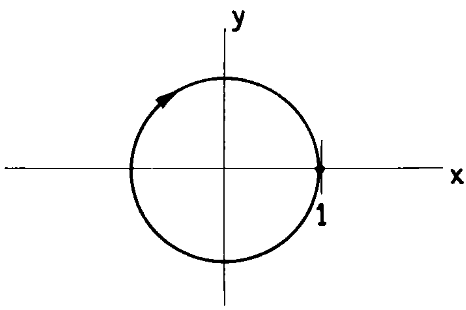

Figure 1.1.3. Orbit through (1,0).

orbit can be thought of as a one parameter family of solutions corresponding to a curve of possible initial conditions for different solutions at a specific time. In the qualitative theory of ordinary differential equations it is not unusual to use the terms orbit and solution interchangeably and, usually, no harm comes from this.

There is a difference in the nature of solutions depending upon whether or not the vector field depends explicitly on the independent variable (note: we will henceforth always refer to the independent variable as time). If the vector field is