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FUNCTIONS OF A COMPLEX VARIABLE
WITH APPLICATIONS

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UNIVERSITY MATHEMATICAL TEXTS

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PREFACE TO THE EIGHTH EDITION

CHANGES that have been made in recent editions include a set of Miscellaneous Examples at the end of the book and an independent proof of Liouville's theorem has been given. In this edition, the proof of the Example on page 50 has been altered.

Limitations of space made it necessary for me to confine myself to the more essential aspects of the theory and its applications, but I have aimed at including those parts of the subject which are most useful to Honours students. Many readers may desire to extend their knowledge of the subject beyond the limits of the present book. Such readers are recommended to study the standard treatises of Copson, *Functions of a Complex Variable* (Oxford, 1935), and Titchmarsh, *Theory of Functions* (Oxford, 1939). I take this opportunity of acknowledging my constant indebtedness to these works both in material and presentation.

I have presupposed a knowledge of Real Variable Theory corresponding approximately to the content of my *Course of Analysis* (Cambridge, Second Edition, 1939). References are occasionally given to this book in footnotes as P.A.

I wish to express my thanks to all those friends who have made helpful suggestions. In particular, I mention two of my colleagues, Mr A. C. Stevenson, of University College, London, who read the proofs of the first edition, and Prof. H. Davenport, F.R.S., who very kindly suggested a number of improvements for the second edition. I desire also to express my gratitude to the publishers for the careful and efficient way in which they have carried out their duties.

E. G. P.

BANGOR, *October* 1956

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CHAPTER I

FUNCTIONS OF A COMPLEX VARIABLE

§ 1. Complex Numbers

This book is concerned essentially with the application of the methods of the differential and integral calculus to complex numbers. A number of the form $\alpha+i\beta$, where i is $\sqrt{(-1)}$ and α and β are real numbers, is called a **complex number**; and, although complex numbers are capable of a geometrical interpretation, it is important to give a *definition* of them which depends only on real numbers. Complex numbers first became necessary in the study of algebraic equations. It is desirable to be able to say that every quadratic equation has two roots, every cubic equation has three roots, and so on. If real numbers only are considered, the equation $x^2+1 = 0$ has no roots and $x^3-1 = 0$ has only one. Every generalisation of number first presented itself as needed for some simple problem, but extensions of number are not created by the mere need of them; they are created by the definition, and our object is now to *define* complex numbers.

By choosing one of several possible lines of procedure, we *define a complex number as an ordered pair of real numbers*. Thus $(4, 3)$, $(\sqrt{2}, e)$, $(\frac{1}{4}, \pi)$ are complex numbers. If we write $z = (x, y)$, x is called the **real part,** and y the **imaginary part,** of the complex number z.

(i) Two complex numbers are **equal** if, and only if, their real and imaginary parts are separately equal. The equation $z = z'$ implies that both $x = x'$ and $y = y'$.

(ii) The **modulus** of z, written $|z|$, is defined to be $+\sqrt{(x^2+y^2)}$. It follows immediately from the definition that $|z| = 0$ if, and only if, $x = 0$ and $y = 0$.

(iii) *The fundamental operations.*

If $z = (x, y)$, $z' = (x', y')$ we have the following *definitions*:

(1) $z+z'$ is $(x+x', y+y')$.
(2) $-z$ is $(-x, -y)$.
(3) $z-z' = z+(-z')$ is $(x-x', y-y')$.
(4) zz' is $(xx'-yy', xy'+x'y)$.

If the fundamental operations are thus defined, we easily see that the fundamental laws of algebra are all satisfied.

(a) *The commutative and associative laws of addition hold:*

$$z_1+z_2 = z_2+z_1;$$
$$z_1+(z_2+z_3) = (z_1+z_2)+z_3 = z_1+z_2+z_3.$$

(b) *The same laws of multiplication hold:*

$$z_1z_2 = z_2z_1;$$
$$z_1(z_2z_3) = (z_1z_2)z_3 = z_1z_2z_3.$$

(c) *The distributive law holds:*

$$(z_1+z_2)z_3 = z_1z_3+z_2z_3.$$

As an example of the method, we show that the commutative law of multiplication holds. The others are proved similarly.

$$z_1z_2 = (x_1x_2-y_1y_2, x_1y_2+x_2y_1)$$
$$= (x_2x_1-y_2y_1, x_2y_1+x_1y_2) = z_2z_1.$$

We have thus seen that complex numbers, as defined above, obey the fundamental laws of the algebra of real numbers: hence their algebra will be identical in *form*, though not in *meaning*, with the algebra of real numbers.

We observe that there is no order among complex numbers. As applied to complex numbers, the phrases "greater than" or "less than" have no meaning. Inequalities can only occur in relations between the *moduli* of complex numbers.

(iv) *The definition of division.*

Consider the equation $z\zeta = z'$, where $z = (x, y)$, $\zeta = (\xi, \eta)$, $z' = (x', y')$, then we have

$$(x\xi - y\eta, x\eta + y\xi) = (x', y'),$$

so that
$$x\xi - y\eta = x', \; x\eta + y\xi = y',$$

and, on solving for ξ and η,

$$\xi = \frac{yy' + xx'}{x^2 + y^2}, \; \eta = \frac{xy' - x'y}{x^2 + y^2};$$

provided that $|z| \neq 0$. Hence, if $|z| \neq 0$, there is a unique solution, and $\zeta = (\xi, \eta)$ is the quotient z'/z.

Division by a complex number whose modulus is zero is meaningless; this conforms with the algebra of real numbers, in which division by zero is meaningless.

The abbreviated notation.

It is customary to denote a complex number whose imaginary part is zero by the real-number symbol x. If we adopt this practice, it is essential to realise that x may have two meanings (i) the real number x, and (ii) the complex number $(x, 0)$. Although in theory it is important to distinguish between (i) and (ii), in practice it is legitimate to confuse them; and if we use the abbreviated notation, in which x stands for $(x, 0)$ and y for $(y, 0)$, then

$$x + y = (x, 0) + (y, 0) = (x + y, 0),$$
$$xy = (x, 0) \,.\, (y, 0) = (x \,.\, y - 0 \,.\, 0, x \,.\, 0 + 0 \,.\, y) = (xy, 0).$$

Hence, so far as sums and products are concerned, complex numbers whose imaginary parts are zero can be treated as though they were real numbers. It is customary to denote the complex number $(0, 1)$ by i. With this convention, $i^2 = (0, 1) \,.\, (0, 1) = (-1, 0)$, so that i may be regarded as the square root of the real number -1. On using the abbreviated notation, it follows that

$$(x, y) = x + iy,$$

for, since $i = (0, 1)$, we have

$$\begin{aligned} x+iy &= (x, 0)+(0, 1) \cdot (y, 0) \\ &= (x, 0)+(0 \cdot y-1 \cdot 0, 0 \cdot 0+1 \cdot y) \\ &= (x, 0)+(0, y) = (x+0, 0+y) = (x, y). \end{aligned}$$

In virtue of this relation we see that, in any operation involving sums and products, it is allowable to treat x, y and i as though they were ordinary real numbers, with the proviso that i^2 must always be replaced by -1.

§ 2. Conjugate Complex Numbers

If $z = x+iy$, it is customary to write $x = \mathbf{R}z$, $y = \mathbf{I}z$. The number $x-iy$ is said to be **conjugate** to z and is usually denoted by $\bar{z}$. It readily follows that the numbers conjugate to z_1+z_2 and z_1z_2 are $\bar{z}_1+\bar{z}_2$ and $\bar{z}_1\bar{z}_2$ respectively.

Proofs of theorems on complex numbers are often considerably simplified by the use of conjugate complex numbers, in virtue of the relations, easily proved,

$$|z|^2 = z\bar{z}, \quad 2\mathbf{R}z = z+\bar{z}, \quad 2i\mathbf{I}z = z-\bar{z}.$$

To prove that *the modulus of the product of two complex numbers is the product of their moduli*, we proceed as follows:

$$|z_1z_2|^2 = z_1z_2\bar{z}_1\bar{z}_2 = z_1\bar{z}_1 \cdot z_2\bar{z}_2 = |z_1|^2 \cdot |z_2|^2$$

and so, since the modulus of a complex number is never negative,

$$|z_1z_2| = |z_1| \cdot |z_2|.$$

Theorem. The modulus of the sum of two complex numbers cannot exceed the sum of their moduli.

$$\begin{aligned} |z_1+z_2|^2 &= (z_1+z_2)(\bar{z}_1+\bar{z}_2) \\ &= z_1\bar{z}_1+z_1\bar{z}_2+\bar{z}_1z_2+z_2\bar{z}_2 \\ &= |z_1|^2+2\mathbf{R}(z_1\bar{z}_2)+|z_2|^2 \\ &\leqslant |z_1|^2+2|z_1\bar{z}_2|+|z_2|^2 \\ &= (|z_1|+|z_2|)^2, \end{aligned}$$

and so $$|z_1+z_2| \leqslant |z_1|+|z_2|;$$

a result which can be readily extended by induction to any finite number of complex numbers.

In a similar way we can prove another useful result, viz.

$$| z_1 - z_2 | \geqslant | (| z_1 | - | z_2 |) |.$$

We have

$$\begin{aligned} | z_1 - z_2 |^2 &= | z_1 |^2 - 2\mathbf{R}(z_1 \bar{z}_2) + | z_2 |^2 \\ &\geqslant | z_1 |^2 - 2 | z_1 \bar{z}_2 | + | z_2 |^2 \\ &= (| z_1 | - | z_2 |)^2 ; \end{aligned}$$

hence $$| z_1 - z_2 | \geqslant |(| z_1 | - | z_2 |)|.$$

§ 3. Geometrical Representation of Complex Numbers

If we denote $(x^2+y^2)^{\frac{1}{2}}$ by r, and choose θ so that $r \cos \theta = x$, $r \sin \theta = y$, then r and θ are clearly the radius

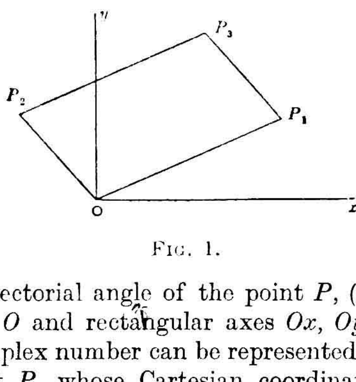

FIG. 1.

vector and vectorial angle of the point P, (x, y), referred to an origin O and rectangular axes Ox, Oy. It is clear that any complex number can be represented geometrically by the point P, whose Cartesian coordinates are (x, y) or whose polar coordinates are (r, θ), and the representation of complex numbers thus afforded is called the **Argand diagram.**

By the definition already given, it is evident that r is the modulus of $z = (x, y)$; the angle θ is called the **argument** of z, written $\theta = \arg z$. The argument is not unique, for if θ be a value of the argument, so also is

$2n\pi+\theta, (n = 0, \pm 1, \pm 2, \ldots)$. The **principal value** of $\arg z$ is that which satisfies the inequalities $-\pi < \arg z \leqslant \pi$.

Let P_1 and P_2 (in fig. 1) be the points z_1 and z_2, then we can represent *addition* in the following way. Through P_1, draw P_1P_3 equal to, and parallel to OP_2. Then P_3 has coordinates (x_1+x_2, y_1+y_2) and so P_3 represents the point z_1+z_2.

In vectorial notation,

$$\overline{OP}_3 = \overline{OP}_1 + \overline{P_1P}_3 = \overline{OP}_1 + \overline{OP}_2 = \overline{OP}_2 + \overline{P_2P}_3.$$

Similarly, we have, if P_3 is the point z_3,

$$z_3 - z_2 = \overline{OP}_3 - \overline{OP}_2 = \overline{OP}_3 + \overline{P_3P}_1 = \overline{OP}_1 = z_1.$$

It is convenient to write $\operatorname{cis}\theta$ for $\cos\theta + i\sin\theta$. If $z_1 = r_1 \operatorname{cis}\theta_1$, $z_2 = r_2 \operatorname{cis}\theta_2$, ..., $z_n = r_n \operatorname{cis}\theta_n$, then, by de Moivre's theorem,

$$z_1 z_2 \ldots z_n = r_1 r_2 \ldots r_n \operatorname{cis}(\theta_1 + \theta_2 + \ldots + \theta_n),$$

which readily exhibits the fact that the modulus and argument of a product are equal respectively to the product of the moduli and the sum of the arguments of the factors. In particular, if n be a positive integer and $z = r \operatorname{cis}\theta$, $z^n = r^n \operatorname{cis} n\theta$.

Similarly,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2).$$

If n is a positive integer, there are n distinct values of $z^{1/n}$. If m is any integer, since

$$\left(\operatorname{cis}\frac{\theta+2m\pi}{n}\right)^n = \operatorname{cis}\theta,$$

it follows that $r^{1/n} \operatorname{cis}\{(\theta+2m\pi)/n\}$ is an nth root of $z = r\operatorname{cis}\theta$. If we substitute the numbers 0, 1, 2, ... $n-1$ in succession for m, we obtain n distinct values of $z^{1/n}$; and the substitution of other integers for m merely gives rise to repetitions of these values. Also, there can be no other values, since $z^{1/n}$ is a root of the equation $u^n = z$ which cannot have more than n roots.

Similarly, if p and q are integers prime to each other and q is positive,

$$z^{p/q} = r^{p/q} \operatorname{cis}\{(p\theta + 2m\pi)/q\},$$

where $m = 0, 1, 2, \ldots, q-1$.

By considering the modulus and argument of a complex number, the operation of multiplying any complex number $x+iy$ by i is easily seen to be equivalent to turning the line OP through a right-angle in the positive (counter-clockwise) sense. We have just seen that

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2, \quad \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2,$$

so that the formal process of "taking arguments" is similar to that of "taking logarithms." Hence, if $\arg(x+iy) = a$,

$$\arg i(x+iy) = \arg i + \arg(x+iy) = \tfrac{1}{2}\pi + a.$$

Since $|i| = 1$, multiplying by i leaves $|x+iy|$ unaltered.

§ 4. Sets of Points in the Argand Diagram

We now explain some of the terminology necessary for dealing with sets of complex numbers in the Argand diagram. We shall use such terms as *domain*, *contour*, *inside and outside of a closed contour*, without more precise definition than geometrical intuition requires. The general study of such questions as the precise determination of the *inside* and *outside* of a closed contour is not so easy as our intuitions might lead us to expect.* For our present purpose, however, we shall find that no difficulties arise from our relying upon geometrical intuition.

By a **neighbourhood** of a point z_0 in the Argand diagram, we mean the set of all points z such that $|z-z_0| < \epsilon$, where ϵ is a given positive number. A point z_0 is said

* For further information, see *e.g.* Dienes, *The Taylor Series* (Oxford, 1931), Ch. VI.

to be a **limit point** of a set of points S, if every neighbourhood of z_0 contains a point of S other than z_0. The definition implies that every neighbourhood of a limit point z_0 contains an infinite number of points of S. For, the neighbourhood $|z-z_0|<\epsilon$ contains a point z_1 of S distinct from z_0, the neighbourhood $|z-z_0|<|z_1-z_0|$ contains a point z_2 of S distinct from z_0 and so on indefinitely.

The limit points of a set are not necessarily points of the set. If, however, every limit point of the set belongs to the set, we say that the set is **closed.** There are two types of limit points, *interior points* and *boundary points.* A limit point z_0 of S is an **interior point** if there exists a neighbourhood of z_0 which consists entirely of points of S. A limit point which is not an interior point is a **boundary point.**

A set which consists entirely of interior points is said to be an **open set.**

It should be observed that a set need not be either open or closed. An example of such a set is that consisting of the point $z = 1$ and all the points for which $|z| < 1$.

We now define a *Jordan curve.*

The equation $z = x(t)+iy(t)$, where $x(t)$ and $y(t)$ are real continuous functions of the real variable t, defined in the range $\alpha \leqslant t \leqslant \beta$, determines a set of points in the Argand diagram which is called a **continuous arc.** A point z_1 is a **multiple point** of the arc, if the equation $z_1 = x(t)+iy(t)$ is satisfied by more than one value of t in the given range.

A continuous arc without multiple points is called a **Jordan arc.** If the points corresponding to the values α and β coincide, the arc, which has only one multiple point, a double point corresponding to the terminal values α and β of t, is called a **simple closed Jordan curve.**

A set of points is said to be **bounded** if there exists a constant K such that $|z| \leqslant K$ is satisfied for all points