Electrodynamics of Continuous Media

2nd edition

Landau and Lifshitz Course of Theoretical Physics

Volume 8

ELECTRODYNAMICS OF CONTINUOUS MEDIA

by

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PREFACE TO THE FIRST ENGLISH EDITION

THE present volume in the *Course of Theoretical Physics* deals with the theory of electromagnetic fields in matter and with the theory of the macroscopic electric and magnetic properties of matter. These theories include a very wide range of topics, as may be seen from the Contents.

In writing this book we have experienced considerable difficulties, partly because of the need to make a selection from the extensive existing material, and partly because the customary exposition of many topics to be included does not possess the necessary physical clarity, and sometimes is actually wrong. We realize that our own treatment still has many defects, which we hope to correct in future editions.

We are grateful to Professor V. L. Ginzburg, who read the book in manuscript and made some useful comments. I. E. Dzyaloshinskiĭ and L. P. Pitaevskiĭ gave great help in reading the proofs of the Russian edition. Thanks are due also to Dr Sykes and Dr Bell, who not only carried out excellently the arduous task of translating the book, but also made some useful comments concerning its contents.

Moscow
June, 1959

L. D. LANDAU
E. M. LIFSHITZ

NOTATION

Electric field **E**

Electric induction **D**

Magnetic field **H**

Magnetic induction **B**

External electric field $\mathfrak{E}$, magnitude $\mathfrak{E}$

External magnetic field $\mathfrak{H}$, magnitude $\mathfrak{H}$

Dielectric polarization **P**

Magnetization **M**

Total electric moment of a body $\mathscr{P}$

Total magnetic moment of a body $\mathscr{M}$

Permittivity ε

Dielectric susceptibility κ

Magnetic permeability μ

Magnetic susceptibility χ

Current density **j**

Conductivity σ

Absolute temperature (in energy units) T

Pressure P

Volume V

Thermodynamic quantities:	per unit volume	for a body
entropy	S	$\mathscr{S}$
internal energy	U	$\mathscr{U}$
free energy	F	$\mathscr{F}$
thermodynamic potential (Gibbs free energy)	Φ	$\mathscr{G}$

Chemical potential ζ

A complex periodic time factor is always taken as $e^{-i\omega t}$.

Volume element dV or d^3x; surface element **df**.

The summation convention always applies to three-dimensional (Latin) and two-dimensional (Greek) suffixes occurring twice in vector and tensor expressions.

References to other volumes in the *Course of Theoretical Physics*.

Mechanics = Vol. 1 (*Mechanics*, third English edition, 1976).

Fields = Vol. 2 (*The Classical Theory of Fields*, fourth English edition, 1975).

QM = Vol. 3 (*Quantum Mechanics—Non-relativistic theory*, third English edition, 1977).

QED = Vol. 4 (*Quantum Electrodynamics*, second English edition, 1982).

SP 1 = Vol. 5 (*Statistical Physics*, Part 1, third English edition, 1980).

FM = Vol. 6 (*Fluid Mechanics*, English edition, 1959).

TE = Vol. 7 (*Theory of Elasticity*, second English edition, 1970).

SP 2 = Vol. 9 (*Statistical Physics*, Part 2, English edition, 1980).

PK = Vol. 10 (*Physical Kinetics*, English edition, 1981).

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CONTENTS

CHAPTER I

ELECTROSTATICS OF CONDUCTORS

§1. The electrostatic field of conductors

Macroscopic electrodynamics is concerned with the study of electromagnetic fields in space that is occupied by matter. Like all macroscopic theories, electrodynamics deals with physical quantities averaged over elements of volume which are "physically infinitesimal", ignoring the microscopic variations of the quantities which result from the molecular structure of matter. For example, instead of the actual "microscopic" value of the electric field e, we discuss its averaged value, denoted by **E**:

$$\bar{\mathbf{e}} = \mathbf{E}. \tag{1.1}$$

The fundamental equations of the electrodynamics of continuous media are obtained by averaging the equations for the electromagnetic field in a vacuum. This method of obtaining the macroscopic equations from the microscopic was first used by H. A. Lorentz (1902).

The form of the equations of macroscopic electrodynamics and the significance of the quantities appearing in them depend essentially on the physical nature of the medium, and on the way in which the field varies with time. It is therefore reasonable to derive and investigate these equations separately for each type of physical object.

It is well known that all bodies can be divided, as regards their electric properties, into two classes, *conductors* and *dielectrics*, differing in that any electric field causes in a conductor, but not in a dielectric, the motion of charges, i.e. an *electric current*.†

Let us begin by studying the static electric fields produced by charged conductors, that is, the *electrostatics of conductors*. First of all, it follows from the fundamental property of conductors that, in the electrostatic case, the electric field inside a conductor must be zero. For a field **E** which was not zero would cause a current; the propagation of a current in a conductor involves a dissipation of energy, and hence cannot occur in a stationary state (with no external sources of energy).

Hence it follows, in turn, that any charges in a conductor must be located on its surface. The presence of charges inside a conductor would necessarily cause an electric field in it;‡ they can be distributed on its surface, however, in such a way that the fields which they produce in its interior are mutually balanced.

Thus the problem of the electrostatics of conductors amounts to determining the electric field in the vacuum outside the conductors and the distribution of charges on their surfaces.

At any point far from the surface of the body, the mean field **E** in the vacuum is almost

† The conductor is here assumed to be homogeneous (in composition, temperature, etc.). In an inhomogeneous conductor, as we shall see later, there may be fields which cause no motion of charges.

‡ This is clearly seen from equation (1.8) below.

the same as the actual field $\mathbf{e}$. The two fields differ only in the immediate neighbourhood of the body, where the effect of the irregular molecular fields is noticeable, and this difference does not affect the averaged field equations. The exact microscopic Maxwell's equations in the vacuum are

$$\operatorname{div} \mathbf{e} = 0, \tag{1.2}$$

$$\mathbf{curl}\, \mathbf{e} = -(1/c)\partial \mathbf{h}/\partial t, \tag{1.3}$$

where $\mathbf{h}$ is the microscopic magnetic field. Since the mean magnetic field is assumed to be zero, the derivative $\partial \mathbf{h}/\partial t$ also vanishes on averaging, and we find that the static electric field in the vacuum satisfies the usual equations

$$\operatorname{div} \mathbf{E} = 0, \qquad \mathbf{curl}\, \mathbf{E} = 0, \tag{1.4}$$

i.e. it is a potential field with a potential ϕ such that

$$\mathbf{E} = -\mathbf{grad}\, \phi, \tag{1.5}$$

and ϕ satisfies Laplace's equation

$$\triangle \phi = 0. \tag{1.6}$$

The boundary conditions on the field $\mathbf{E}$ at the surface of a conductor follow from the equation $\mathbf{curl}\, \mathbf{E} = 0$, which, like the original equation (1.3), is valid both outside and inside the body. Let us take the z-axis in the direction of the normal $\mathbf{n}$ to the surface at some point on the conductor. The component E_z of the field takes very large values in the immediate neighbourhood of the surface (because there is a finite potential difference over a very small distance). This large field pertains to the surface itself and depends on the physical properties of the surface, but is not involved in our electrostatic problem, because it falls off over distances comparable with the distances between atoms. It is important to note, however, that, if the surface is homogeneous, the derivatives $\partial E_z/\partial x$, $\partial E_z/\partial y$ along the surface remain finite, even though E_z itself becomes very large. Hence, since $(\mathbf{curl}\, \mathbf{E})_x = \partial E_z/\partial y - \partial E_y/\partial z = 0$, we find that $\partial E_y/\partial z$ is finite. This means that E_y is continuous at the surface, since a discontinuity in E_y would mean an infinity of the derivative $\partial E_y/\partial z$. The same applies to E_x, and since $\mathbf{E} = 0$ inside the conductor, we reach the conclusion that the tangential components of the external field at the surface must be zero:

$$\mathbf{E}_t = 0. \tag{1.7}$$

Thus the electrostatic field must be normal to the surface of the conductor at every point. Since $\mathbf{E} = -\mathbf{grad}\, \phi$, this means that the field potential must be constant on the surface of any particular conductor. In other words, the surface of a homogeneous conductor is an equipotential surface of the electrostatic field.

The component of the field normal to the surface is very simply related to the charge density on the surface. The relation is obtained from the general electrostatic equation $\operatorname{div} \mathbf{e} = 4\pi\rho$, which on averaging gives

$$\operatorname{div} \mathbf{E} = 4\pi\bar{\rho}, \tag{1.8}$$

$\bar{\rho}$ being the mean charge density. The meaning of the integrated form of this equation is well known: the flux of the electric field through a closed surface is equal to the total charge inside that surface, multiplied by 4π. Applying this theorem to a volume element lying between two infinitesimally close unit areas, one on each side of the surface of the conductor, and using the fact that $\mathbf{E} = 0$ on the inner area, we find that $E_n = 4\pi\sigma$, where σ is the surface charge density, i.e. the charge per unit area of the surface of the conductor. Thus the distribution of charges over the surface of the conductor is given by the formula

$$4\pi\sigma = E_n = -\partial\phi/\partial n, \tag{1.9}$$

the derivative of the potential being taken along the outward normal to the surface. The total charge on the conductor is

$$e = -\frac{1}{4\pi}\oint\frac{\partial\phi}{\partial n}\,\mathrm{d}f, \tag{1.10}$$

the integral being taken over the whole surface.

The potential distribution in the electrostatic field has the following remarkable property: the function $\phi(x, y, z)$ can take maximum and minimum values only at boundaries of regions where there is a field. This theorem can also be formulated thus: a test charge e introduced into the field cannot be in stable equilibrium, since there is no point at which its potential energy $e\phi$ would have a minimum.

The proof of the theorem is very simple. Let us suppose, for example, that the potential has a maximum at some point A not on the boundary of a region where there is a field. Then the point A can be surrounded by a small closed surface on which the normal derivative $\partial\phi/\partial n < 0$ everywhere. Consequently, the integral over this surface $\oint(\partial\phi/\partial n)\,\mathrm{d}f < 0$. But by Laplace's equation $\oint(\partial\phi/\partial n)\mathrm{d}f = \int\triangle\phi\,\mathrm{d}V = 0$, giving a contradiction.

§2. The energy of the electrostatic field of conductors

Let us calculate the total energy $\mathscr{U}$ of the electrostatic field of charged conductors,†

$$\mathscr{U} = \frac{1}{8\pi}\int\mathbf{E}^2\mathrm{d}V, \tag{2.1}$$

where the integral is taken over all space outside the conductors. We transform this integral as follows:

$$\mathscr{U} = -\frac{1}{8\pi}\int\mathbf{E}\cdot\mathbf{grad}\,\phi\,\mathrm{d}V = -\frac{1}{8\pi}\int\mathrm{div}\,(\phi\mathbf{E})\,\mathrm{d}V + \frac{1}{8\pi}\int\phi\,\mathrm{div}\,\mathbf{E}\,\mathrm{d}V.$$

The second integral vanishes by (1.4), and the first can be transformed into integrals over the surfaces of the conductors which bound the field and an integral over an infinitely remote surface. The latter vanishes, because the field diminishes sufficiently rapidly at infinity (the arbitrary constant in ϕ is assumed to be chosen so that $\phi = 0$ at infinity). Denoting by ϕ_a the constant value of the potential on the ath conductor, we have‡

$$\mathscr{U} = \frac{1}{8\pi}\sum_a\oint\phi E_n\,\mathrm{d}f = \frac{1}{8\pi}\sum_a\phi_a\oint E_n\,\mathrm{d}f.$$

† The square $\mathbf{E}^2$ is not the same as the mean square $\overline{\mathbf{e}^2}$ of the actual field near the surface of a conductor or inside it (where $\mathbf{E} = 0$ but, of course, $\overline{\mathbf{e}^2} \neq 0$). By calculating the integral (2.1) we ignore the internal energy of the conductor as such, which is here of no interest, and the affinity of the charges for the surface.

‡ In transforming volume integrals into surface integrals, both here and later, it must be borne in mind that E_n is the component of the field along the outward normal to the conductor. This direction is opposite to that of the outward normal to the region of the volume integration, namely the space outside the conductors. The sign of the integral is therefore changed in the transformation.

Finally, since the total charges e_a on the conductors are given by (1.10) we obtain

$$\mathscr{U} = \tfrac{1}{2}\sum_a e_a \phi_a, \tag{2.2}$$

which is analogous to the expression for the energy of a system of point charges.

The charges and potentials of the conductors cannot both be arbitrarily prescribed; there are certain relations between them. Since the field equations in a vacuum are linear and homogeneous, these relations must also be linear, i.e. they must be given by equations of the form

$$e_a = \sum_b C_{ab}\phi_b, \tag{2.3}$$

where the quantities C_{aa}, C_{ab} have the dimensions of length and depend on the shape and relative position of the conductors. The quantities C_{aa} are called *coefficients of capacity*, and the quantities $C_{ab}(a \neq b)$ are called *coefficients of electrostatic induction*. In particular, if there is only one conductor, we have $e = C\phi$, where C is the *capacitance*, which in order of magnitude is equal to the linear dimension of the body. The converse relations, giving the potentials in terms of the charges, are

$$\phi_a = \sum_b C^{-1}{}_{ab} e_b, \tag{2.4}$$

where the coefficients $C^{-1}{}_{ab}$ form a matrix which is the inverse of the matrix C_{ab}.

Let us calculate the change in the energy of a system of conductors caused by an infinitesimal change in their charges or potentials. Varying the original expression (2.1), we have $\delta\mathscr{U} = (1/4\pi)\int \mathbf{E}\cdot\delta\mathbf{E}\,\mathrm{d}V$. This can be further transformed by two equivalent methods. Putting $\mathbf{E} = -\,\mathbf{grad}\,\phi$ and using the fact that the varied field, like the original field, satisfies equations (1.4) (so that $\operatorname{div}\delta\mathbf{E} = 0$), we can write

$$\delta\mathscr{U} = -\frac{1}{4\pi}\int \mathbf{grad}\,\phi\cdot\delta\mathbf{E}\,\mathrm{d}V = -\frac{1}{4\pi}\int \operatorname{div}(\phi\,\delta\mathbf{E})\,\mathrm{d}V$$

$$= \frac{1}{4\pi}\sum_a \phi_a \oint \delta E_n\,\mathrm{d}f,$$

that is

$$\delta\mathscr{U} = \sum_a \phi_a\,\delta e_a, \tag{2.5}$$

which gives the change in energy due to a change in the charges. This result is obvious; it is the work required to bring infinitesimal charges δe_a to the various conductors from infinity, where the field potential is zero.

On the other hand, we can write

$$\delta\mathscr{U} = -\frac{1}{4\pi}\int \mathbf{E}\cdot\mathbf{grad}\,\delta\phi\,\mathrm{d}V = -\frac{1}{4\pi}\int \operatorname{div}(\mathbf{E}\,\delta\phi)\,\mathrm{d}V$$

$$= \frac{1}{4\pi}\sum_a \delta\phi_a \oint E_n\,\mathrm{d}f,$$

that is

$$\delta \mathscr{U} = \sum_a e_a \delta\phi_a, \tag{2.6}$$

which expresses the change in energy in terms of the change in the potentials of the conductors.

Formulae (2.5) and (2.6) show that, by differentiating the energy $\mathscr{U}$ with respect to the charges, we obtain the potentials of the conductors, and the derivatives of $\mathscr{U}$ with respect to the potentials are the charges:

$$\partial \mathscr{U}/\partial e_a = \phi_a, \quad \partial \mathscr{U}/\partial \phi_a = e_a. \tag{2.7}$$

But the potentials and charges are linear functions of each other. Using (2.3) we have $\partial^2 \mathscr{U}/\partial\phi_a\partial\phi_b = \partial e_b/\partial\phi_a = C_{ba}$, and by reversing the order of differentiation we get C_{ab}. Hence it follows that

$$C_{ab} = C_{ba}, \tag{2.8}$$

and similarly $C^{-1}{}_{ab} = C^{-1}{}_{ba}$. The energy $\mathscr{U}$ can be written as a quadratic form in either the potentials or the charges:

$$\mathscr{U} = \tfrac{1}{2}\sum_{a,b} C_{ab}\phi_a\phi_b = \tfrac{1}{2}\sum_{a,b} C^{-1}{}_{ab}e_a e_b. \tag{2.9}$$

This quadratic form must be positive definite, like the original expression (2.1). From this condition we can derive various inequalities which the coefficients C_{ab} must satisfy. In particular, all the coefficients of capacity are positive:

$$C_{aa} > 0 \tag{2.10}$$

(and also $C^{-1}{}_{aa} > 0$).†

All the coefficients of electrostatic induction, on the other hand, are negative:

$$C_{ab} < 0 \quad (a \neq b). \tag{2.11}$$

That this must be so is seen from the following simple arguments. Let us suppose that every conductor except the ath is earthed, i.e. their potentials are zero. Then the charge induced by the charged ath conductor on another (the bth, say) is $e_b = C_{ba}\phi_a$. It is obvious that the sign of the induced charge must be opposite to that of the inducing potential, and therefore $C_{ab} < 0$. This can be more rigorously shown from the fact that the potential of the electrostatic field cannot reach a maximum or minimum outside the conductors. For example, let the potential ϕ_a of the only conductor not earthed be positive. Then the potential is positive in all space, its least value (zero) being attained only on the earthed conductors. Hence it follows that the normal derivative $\partial\phi/\partial n$ of the potential on the surfaces of these conductors is positive, and their charges are therefore negative, by (1.10). Similar arguments show that $C^{-1}{}_{ab} > 0$.

The energy of the electrostatic field of conductors has a certain extremum property, though this property is more formal than physical. To derive it, let us suppose that the

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† We may also mention that another inequality which must be satisfied if the form (2.9) is positive is $C_{aa}C_{bb} > C_{ab}{}^2$.

charge distribution on the conductors undergoes an infinitesimal change (the total charge on each conductor remaining unaltered), in which the charges may penetrate into the conductors; we ignore the fact that such a charge distribution cannot in reality be stationary. We consider the change in the integral $\mathscr{U} = (1/8\pi)\int E^2 \mathrm{d}V$, which must now be extended over all space, including the volumes of the conductors themselves (since after the displacement of the charges the field $\mathbf{E}$ may not be zero inside the conductors). We write

$$\delta\mathscr{U} = -\frac{1}{4\pi}\int \mathbf{grad}\,\phi\cdot\delta\mathbf{E}\,\mathrm{d}V$$

$$= -\frac{1}{4\pi}\int \mathrm{div}\,(\phi\delta\mathbf{E})\,\mathrm{d}V + \frac{1}{4\pi}\int \phi\,\mathrm{div}\,\delta\mathbf{E}\,\mathrm{d}V.$$

The first integral vanishes, being equivalent to one over an infinitely remote surface. In the second integral, we have by (1.8) $\mathrm{div}\,\delta\mathbf{E} = 4\pi\delta\bar{\rho}$, so that $\delta\mathscr{U} = \int\phi\delta\bar{\rho}\,\mathrm{d}V$. This integral vanishes if ϕ is the potential of the true electrostatic field, since then ϕ is constant inside each conductor, and the integral $\int\delta\bar{\rho}\,\mathrm{d}V$ over the volume of each conductor is zero, since its total charge remains unaltered.

Thus the energy of the actual electrostatic field is a minimum† relative to the energies of fields which could be produced by any other distribution of the charges on or in the conductors (*Thomson's theorem*).

From this theorem it follows, in particular, that the introduction of an uncharged conductor into the field of given charges (charged conductors) reduces the total energy of the field. To prove this, it is sufficient to compare the energy of the actual field resulting from the introduction of the uncharged conductor with the energy of the fictitious field in which there are no induced charges on that conductor. The former energy, since it has the least possible value, is less than the latter energy, which is also the energy of the original field (since, in the absence of induced charges, the field would penetrate into the conductor, and remain unaltered). This result can also be formulated thus: an uncharged conductor remote from a system of given charges is attracted towards the system.

Finally, it can be shown that a conductor (charged or not) brought into an electrostatic field cannot be in stable equilibrium under electric forces alone. This assertion generalizes the theorem for a point charge proved at the end of §1, and can be derived by combining the latter theorem with Thomson's theorem. We shall not pause to give the derivation in detail.

Formulae (2.9) are useful for calculating the energy of a system of conductors at finite distances apart. The energy of an uncharged conductor in a uniform external field $\mathfrak{E}$, which may be imagined as due to charges at infinity, requires special consideration. According to (2.2), this energy is $\mathscr{U} = \frac{1}{2}e\phi$, where e is the remote charge which causes the field, and ϕ is the potential at this charge due to the conductor. $\mathscr{U}$ does not include the energy of the charge e in its own field, since we are interested only in the energy of the conductor. The charge on the conductor is zero, but the external field causes it to acquire an electric dipole moment, which we denote by $\mathscr{P}$. The potential of the electric dipole field at a large distance $\mathbf{r}$ from it is $\phi = \mathscr{P}\cdot\mathbf{r}/r^3$. Hence $\mathscr{U} = e\mathscr{P}\cdot\mathbf{r}/2r^3$. But $-e\mathbf{r}/r^3$ is just the field $\mathfrak{E}$ due to the charge e. Thus

$$\mathscr{U} = -\tfrac{1}{2}\mathscr{P}\cdot\mathfrak{E}. \tag{2.12}$$

† We shall not give here the simple arguments which demonstrate that the extremum is a minimum.

Since all the field equations are linear, it is evident that the components of the dipole moment $\mathscr{P}$ are linear functions of the components of the field $\mathfrak{E}$. The coefficients of proportionality between $\mathscr{P}$ and $\mathfrak{E}$ have the dimensions of length cubed, and are therefore proportional to the volume of the conductor:

$$\mathscr{P}_i = V\alpha_{ik}\mathfrak{E}_k, \tag{2.13}$$

where the coefficients α_{ik} depend only on the shape of the body. The quantities $V\alpha_{ik}$ form a tensor, which may be called the *polarizability tensor* of the body. This tensor is symmetrical: $\alpha_{ik} = \alpha_{ki}$, a statement which will be proved in §11. Accordingly, the energy (2.12) is

$$\mathscr{U} = -\tfrac{1}{2}V\alpha_{ik}\mathfrak{E}_i\mathfrak{E}_k. \tag{2.14}$$

PROBLEMS

PROBLEM 1. Express the mutual capacitance C of two conductors (with charges $\pm e$) in terms of the coefficients C_{ab}.

SOLUTION. The *mutual capacitance* of two conductors is defined as the coefficient C in the relation $e = C(\phi_2 - \phi_1)$, and the energy of the system is given in terms of C by $\mathscr{U} = \frac{1}{2}e^2/C$. Comparing with (2.9), we obtain

$$1/C = C^{-1}{}_{11} - 2C^{-1}{}_{12} + C^{-1}{}_{22}$$
$$= (C_{11} + 2C_{12} + C_{22})/(C_{11}C_{22} - C_{12}{}^2).$$

PROBLEM 2. A point charge e is situated at O, near a system of earthed conductors, and induces on them charges e_a. If the charge e were absent, and the ath conductor were at potential ϕ'_a, the remainder being earthed, the field potential at O would be ϕ'_0. Express the charges e_a in terms of ϕ'_a and ϕ'_0.

SOLUTION. If charges e_a on the conductors give them potentials ϕ_a, and similarly for e'_a and ϕ'_a, it follows from (2.3) that

$$\sum_a \phi_a e'_a = \sum_{a,b} \phi_a C_{ab}\phi'_b = \sum_a \phi'_a e_a.$$

We apply this relation to two states of the system formed by all the conductors and the charge e (regarding the latter as a very small conductor). In one state the charge e is present, the charges on the conductors are e_a, and their potentials are zero. In the other state the charge e is zero, and one of the conductors has a potential $\phi'_a \neq 0$. Then we have $e\phi'_0 + e_a\phi'_a = 0$, whence $e_a = -e\phi'_0/\phi'_a$.

For example, if a charge e is at a distance r from the centre of an earthed conducting sphere with radius a ($< r$), then $\phi'_0 = \phi'_a a/r$, and the charge induced on the sphere is $e_a = -ea/r$.

As a second example, let us consider a charge e placed between two concentric conducting spheres with radii a and b, at a distance r from the centre such that $a < r < b$. If the outer sphere is earthed and the inner one is charged to potential ϕ'_a, the potential at distance r is

$$\phi'_0 = \phi'_a \frac{1/r - 1/b}{1/a - 1/b}.$$

Hence the charge induced on the inner sphere by the charge e is $e_a = -ea(b-r)/r(b-a)$. Similarly the charge induced on the outer sphere is $e_b = -eb(r-a)/r(b-a)$.

PROBLEM 3. Two conductors, with capacitances C_1 and C_2, are placed at a distance r apart which is large compared with their dimensions. Determine the coefficients C_{ab}.

SOLUTION. If conductor 1 has a charge e_1, and conductor 2 is uncharged, then in the first approximation $\phi_1 = e_1/C_1$, $\phi_2 = e_1/r$; here we neglect the variation of the field over conductor 2 and its polarization. Thus $C^{-1}{}_{11} = 1/C_1$, $C^{-1}{}_{12} = 1/r$, and similarly $C^{-1}{}_{22} = 1/C_2$. Hence we find†

$$C_{11} = C_1\left(1 + \frac{C_1C_2}{r^2}\right), \quad C_{12} = -\frac{C_1C_2}{r}, \quad C_{22} = C_2\left(1 + \frac{C_1C_2}{r^2}\right).$$

† The subsequent terms in the expansion are in general of order (in $1/r$) one higher than those given. If, however, r is taken as the distance between the "centres of charge" of the two bodies (for spheres, between the geometrical centres), then the order of the subsequent terms is two higher.