PURE AND APPLIED MATHEMATICS
A WILEY-INTERSCIENCE SERIES OF
TEXTS, MONOGRAPHS & TRACTS

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WILEY

A Wiley-Interscience Publication

JOHN WILEY & SONS

New York • Chichester • Brisbane • Toronto • Singapore

***Library of Congress Cataloging-in-Publication Data*:**

Berman, Abraham.
Nonnegative matrices in dynamic systems.
(Pure and applied mathematics)

"A Wiley-Interscience publication."
Bibliography: p.
Includes index.
1. Non-negative matrices. 2. System analysis.
I. Neumann, Michael, 1946– II. Stern, Ronald J.
III. Title.
QA188.B466 1989 512.9'434 88-33934
ISBN 0-471-62074-2

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

PREFACE

This book concerns the application of the theory of nonnegative matrices to certain problems arising in positive linear differential and control systems. As such, the book will be useful as a graduate level course or seminar text in applied mathematics. Furthermore, since nonnegative matrices and positive linear dynamic systems are relevant in economics, engineering, numerical analysis, operations research, as well as various life and social sciences, we expect that it will also be of interest to workers in these fields.

In order to give the reader a feeling for the content of the book, we now give simplified descriptions of some of the applied problems and topics studied. In doing so, we make some use of certain real world dynamic models for expository purposes. These models and others are described in greater detail in the glossary of Section 3.4; the models are used in the book in order to provide simple illustrations of various aspects of the theory being developed.

1. THE LINEAR DIFFERENTIAL SYSTEM $\dot{x}(t) = Ax(t)$

Here A is a real constant $n \times n$ matrix.

1.1. Exponential Nonnegativity

Given a proper cone $\mathcal{K} \subseteq R^n$, determine whether $x(0) \in \mathcal{K} \Rightarrow x(t) \in \mathcal{K}$ for all $t \geq 0$. In other words, we wish to characterize the situation wherein every initial state $x(0)$ in $\mathcal{K}$ gives rise to a solution $x(t) = e^{tA}x(0)$, which remains in $\mathcal{K}$ for all time. Should this occur, we say that A is *exponentially $\mathcal{K}$-nonnegative*. For an illustration of this idea, consider an n-species ecological system, where $x_i(t)$ represents the mass of species i at time t, and a_{ij} is a coefficient which reflects the effect of the mass of species j on the rate of change of species i. Then the diagonal terms a_{ii} represent the difference between the birth and death rates of species i; there is no sign restriction on this difference. Since no species can have negative mass, a reasonable requirement on the dynamics of this model is that $x(0) \geq 0 \Rightarrow x(t) \geq 0$ for all

$t \geq 0$; that is, we require exponential R^n_+-nonnegativity of A. It will be seen that this is equivalent to all the off-diagonal entries of A being nonnegative. We call this the *species cooperation condition*, since it says the system is in symbiosis; that is, no species has a detrimental effect on any other species.

1.2. Cone Reachability

For a proper cone $\mathcal{K} \subseteq R^n$ and an exponentially $\mathcal{K}$-nonnegative matrix A, find the set of initial states $x(0) \in R^n$ such that $x(t) \in \mathcal{K}$ for all sufficiently large t. In other words, we wish to characterize the set of initial states $X_A(\mathcal{K})$ which give rise to solutions of $\dot{x}(t) = Ax(t)$, which enter and remain in the cone $\mathcal{K}$. Again consider the symbiotic model described above. Given an initial state $x(0) \geq 0$, suppose that we wish to determine whether $\dot{x}(t) = e^{tA}Ax(0) \geq 0$ for all t sufficiently large; that is, all species eventually have nondecreasing mass. This occurs if and only if $Ax(0) \in X_A(R^n_+)$.

1.3. Extended M-Matrices

Given a proper cone $\mathcal{K} \subseteq R^n$, suppose that the matrix $-A$ is *essentially $\mathcal{K}$-nonnegative*; that is, $(\alpha I - A)\mathcal{K} \subseteq \mathcal{K}$ for some real number α. Then A is said to be an *M-matrix with respect to $\mathcal{K}$* provided that no eigenvalue of A has a strictly negative real part; M-matrices are important in economics, mathematical programming, statistics, and numerical analysis, especially for $\mathcal{K} = R^n_+$. We shall investigate how various aspects of the theory of M-matrices generalize when the assumption of essential $\mathcal{K}$-nonnegativity is replaced by the generally weaker assumption of exponential $\mathcal{K}$-nonnegativity.

2. THE POSITIVE LINEAR CONTROL SYSTEM $\dot{x}(t) = Ax(t) + Bu(t)$

Here A and B are real constant $n \times n$ and $n \times m$ matrices, respectively. The time varying vectors $x(t)$ and $u(t)$ are called the *state* and *control* (or *input*) functions, respectively. By a "positive" control system we mean one in which either the control, the state, or some system output vector might be required to belong to a given proper cone.

2.1. Controllability with Nonnegative Controls

Given that the control function is required to be valued in a proper cone for each $t \geq 0$, determine whether the origin 0 is controllable; that is, whether there exists an open neighborhood $\mathcal{N}_o$ of the origin such that for any initial state $x(0) \in \mathcal{N}_o$ there exists an admissible control function such that $x(\hat{t}) = 0$ for some $\hat{t} > 0$. As an illustration, consider the harmonic oscillator given by the spring-mass system shown below.

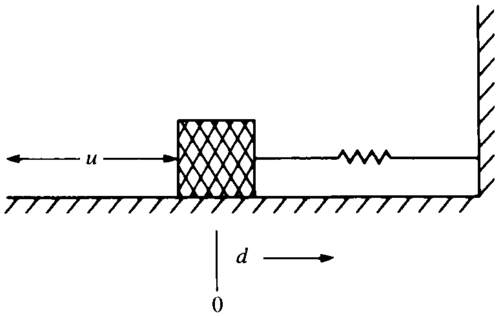

Figure P-1

Here $d(t)$ denotes the displacement from the neutral position of the spring, and $u(t)$ is a horizontally applied force. The state of this system is $x(t) = [d(t), \dot{d}(t)]$. It will be seen that $(0, 0)$ is controllable with nonnegative controls provided that the coefficient of friction is sufficiently large relative to the spring constant. In other words, when the physical constants satisfy this condition, the system can be steered to rest by "pushing only to the right," for any initial state in an open neighborhood of $(0, 0)$.

2.2. Observability with Conical Observation Set

For a proper cone $\mathcal{K} \subseteq R^n$ and a $q \times n$ matrix C, we say that the control system given by $\dot{x}(t) = Ax(t) + Bu(t)$ is *$\mathcal{K}$-observable* provided that the following holds. If $Cx(t)$ and $C\tilde{x}(t)$ are two output vectors which arise from the same input and whose difference is contained in $\mathcal{K}$ on a nontrivial time interval $[0, T]$, then $x(t) = \tilde{x}(t)$ for all $t \geq 0$. As an illustration, consider the following electrical circuit.

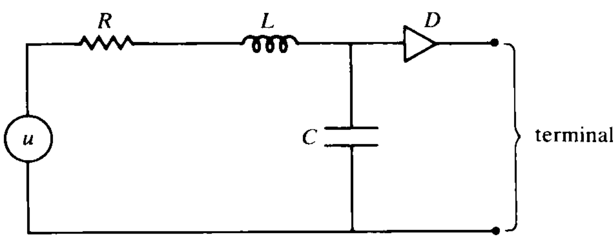

Figure P-2

Here R, L, and C denote resistance, inductance, and capacitance, respectively; D is a diode; and $u(t)$ is the applied voltage. The state of this system is $[x_1(t), x_2(t)]$, where x_1 denotes the current in the loop and x_2 denotes the voltage across the terminal. Taking $C = (0, 1)$ and $\mathcal{K} = R_+$, this system will be seen to be $\mathcal{K}$-observable. This implies that if the initial state is not the origin, then the voltage will at some future time be negative if no control is applied.

2.3. Positive Realization

For a given scalar input–scalar output map $\mathcal{F}$, find a *positive realization.* That is, find a control system $\dot{x}(t) = Ax(t) + Bu(t)$ (where the control is scalar valued) and a column vector c such that any nonnegative control function gives rise to a nonnegative output $c^T x(t) = (\mathcal{F}[u])(t)$.

2.4. Controllability to R^n_+

Find the set of initial states $x(0) \in R^n$ such that by suitable choice of control in the system $\dot{x}(t) = Ax(t) + Bu(t)$ we have $x(t) \geq 0$ for all t sufficiently large. Upon applying results on exponentially nonnegative matrices in certain cases, we will find that the R^n_+-controllability problem reduces to characterizing the existence of an $m \times n$ matrix F such that $A + BF$ is essentially R^n_+-nonnegative and irreducible. It will also be seen that under the feedback law $u(t) = Fx(t)$, any nonnegative initial state gives rise to a trajectory $x(t)$ all of whose components are strictly positive for all positive time.

2.5. Stabilizability-holdability

Find, if possible, a linear feedback law $u(t) = Fx(t)$ such that under this law all trajectories of the control system $\dot{x}(t) = Ax(t) + Bu(t)$ which originate in R^n_+ remain in R^n_+ for all $t \geq 0$ while asymptotically deteriorating to the origin. This problem is equivalent to finding an $m \times n$ matrix F such that $-(A + BF)$ is a nonsingular M-matrix with respect to R^n_+.

The book consists of eight chapters. The first two are concise reviews of the requisite material in convex analysis and matrix theory, respectively. (The bibliographic notes to Chapters 1 and 2 provide references where the proofs of the main results may be found.) Chapter 3 contains a detailed review of linear differential and control systems. The remaining five chapters address the applied subjects touched upon above, and related topics. The book could serve as the text for either a one or two semester graduate level course or seminar, depending on how much time is devoted to the background material.

The logical interdependence of the chapters is summarized in Figure P-3.

The book contains exercises whose purpose is to reinforce the background and to strengthen understanding of the applications. We recommend that all the exercises be attempted.

Internal referencing makes use of the following protocols: Lemma 3.47 refers to Lemma 3.47 of the same chapter, while Lemma 3.3.47 refers to

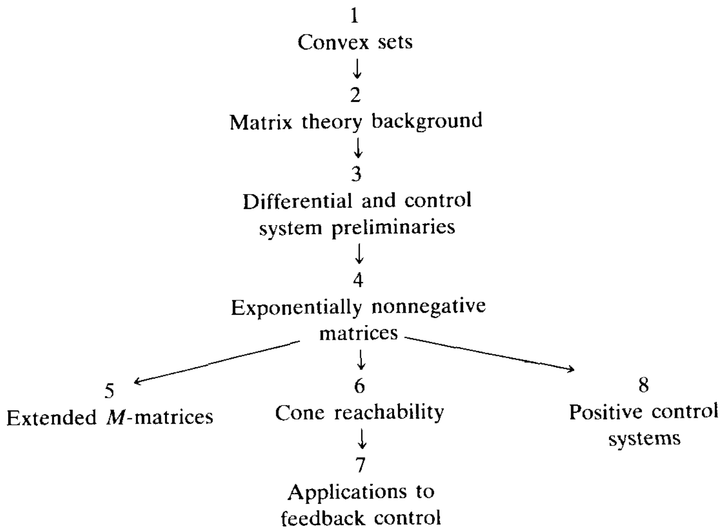

Figure P-3

Lemma 3.47 of Chapter 3. The same method of numbering also applies to other items such as definitions, exercises, etc. In the bibliographic notes at the end of each chapter, we utilize the following conventions: Jones [1981] refers to a publication by Jones which appeared in 1981. A reference such as [1977a] or [1977b] indicates multiple references in 1977 by an author. Also, references such as Smith [a] refer to work which has not yet appeared in print at the time that the manuscript of this book was submitted for publication.

ACKNOWLEDGMENTS

This book was written during the author's work in the mathematics departments at the following universities: The Technion-Israel Institute of Technology, The University of Connecticut, Concordia University, l'Université de Montréal, The University of California at San Diego, and The University of California at Santa Barbara. We wish to thank our colleagues at these and other institutes for their interest and helpful comments.

The authors would like to extend their special thanks to Professor Hans Schneider for reading parts of the book, for his encouragement of the authors, and for providing the proof of Lemma 6.4.32.

During the writing of this book, we were generously financially supported by granting agencies in Israel, the United States, and Canada. The first author was supported by the Fund for Promotion of Research at the Technion and by the Technion VPR Research Fund. The second author received financial support from the National Science Foundation (Grant DMS870064) and the United States Air Force (Grant AFOSR-88-0047), while the third author's support came from the Natural Sciences and Engineering Research Council Canada (Grant A4641).

Finally, we are deeply indebted to Maria Taylor, our editor at Wiley-Interscience, for her patience and encouragement during the preparation of this book.

LIST OF SYMBOLS

$\|A\|$	multiplicative matrix norm subordinate to the euclidean vector norm
$A^{\#}$	group inverse
A^{D}	Drazin generalized inverse
$\{A, B, \Omega\}$	linear control system
$\{A, B, C\}$	linear observed process
$A \geq 0$	nonnegativity of A
$A \overset{e}{\geq} 0$	essential nonnegativity of A
$A \gg 0$	positivity of A
$A \overset{e}{\gg} 0$	essential positivity of A
$\mathrm{aff}\,(W)$	affine hull of W
$\mathscr{A}(x, A, B, \Omega)$	attainable set from x
$A_T(x, A, B, \Omega)$	attainable set from x in time T
$[B : A]$	the matrix $[B \quad AB \quad \ldots \quad A^{n-1}B]$
$\mathrm{cl}\,(W)$	closure of W
$C^{n \times n}$	set of $n \times n$ complex matrices
$\mathscr{C}(\Gamma, A, B, \Omega)$	controllable set to Γ
$\hat{\mathscr{C}}(\Gamma, \mathrm{A}, \mathrm{B}, \Omega)$	strongly controllable set to Γ
$\mathscr{C}_{\mathrm{T}}(\Gamma, A, B, \Omega)$	controllable set to Γ in time T
$\mathrm{cone}\,(W)$	conical hull
$\mathrm{cone}\,[Q]$	cone generated by the columns of the matrix Q
$\mathrm{conv}\,(W)$	convex hull of W
$\mathrm{core}_A(\Gamma)$	core of Γ with respect to A
$d(\lambda)$	degree of the eigenvalue λ
$d(x, W)$	distance from point to set
$\det A$	determinant
∂W	boundary of W
$\mathrm{EP}\,(W)$	set of extreme points of W

$G(A)$	directed graph of A
$h(V, W)$	Hausdorff distance
index (A)	index of A
int W	interior of W
$J(A)$	Jordan form of A
$J(\lambda; k)$	Jordan block of order k
$\bar{\lambda}$	complex conjugate
$\lambda(A)$	spectral abscissa of A
$\mathcal{N}(A)$	nullspace of A
$\mathcal{N}_\lambda$	generalized eigenspace
$\mathcal{N}_W(x)$	set of outward unit normals to W at x
P_λ	complementary eigenprojection
$p(\mathcal{K})$	set of exponentially $\mathcal{K}$-nonnegative matrices
$p'(\mathcal{K})$	set of almost exponentially $\mathcal{K}$-irreducible matrices
$p''(\mathcal{K})$	set of exponentially $\mathcal{K}$-irreducible matrices
$p^+(\mathcal{K})$	set of exponentially $\mathcal{K}$-positive matrices
$\pi(\mathcal{K})$	set of $\mathcal{K}$-nonnegative matrices
$\pi'(\mathcal{K})$	set of $\mathcal{K}$-irreducible matrices
$\pi^+(\mathcal{K})$	set of $\mathcal{K}$-positive matrices
$r(\mathcal{K})$	set of essentially $\mathcal{K}$-nonnegative matrices
$r'(\mathcal{K})$	set of essentially $\mathcal{K}$-irreducible matrices
$r^+(\mathcal{K})$	set of essentially $\mathcal{K}$-positive matrices
$R^{n \times n}$	set of $n \times n$ real matrices
$R(A)$	reduced graph of A
$\mathcal{R}(A)$	range of A
$R(\lambda, \bar{\lambda}; 2k)$	real canonical block of order $2k$
$R(\lambda; k)$	real canonical block of order k
rank (A)	rank of A
$\mathrm{rb}_Q(W)$	boundary of W relative to an affine set Q
$RC(A)$	real canonical form of A
$\mathrm{ri}_Q(W)$	interior of W relative to an affine set Q
$\rho(A)$	spectral radius of A
$\mathcal{S}_A$	$\bigcap_{m=0}^{\infty} \mathcal{R}(A^m)$, where $A \in C^{n \times n}$
$\mathcal{S}(A, C)$	0-observation set
$\mathcal{S}(A, C, \mathcal{K})$	$\mathcal{K}$-observation set
spec (A)	spectrum of A
sub $(\mathcal{K})$	set of $\mathcal{K}$-subtangential matrices
sub$'(\mathcal{K})$	set of strongly $\mathcal{K}$-subtangential matrices

$\mathrm{sub}^+(\mathcal{K})$	set of strictly $\mathcal{K}$-subtangential matrices
$\mathcal{U}_\Omega[a, b]$	set of essentially bounded Lebesgue measurable functions u: $[a, b] \to \Omega$
$\mathcal{U}_\Omega$	class of admissible control functions
$\mathcal{V}^A$	maximal A-invariant subspace of $\mathcal{V}$
W^*	dual cone of W
$\mathcal{W}_\lambda$	real generalized eigenspace
$\|x\|$	euclidean norm of the vector x
$X_A(\mathcal{K})$	reachability cone for $\mathcal{K}$ under A
$X_{A,h}(R^n_+)$	discrete reachability cone for R^n_+ under A

CONTENTS

1

CONVEX SETS

1. INTRODUCTION

This chapter constitutes a review of the material on convexity which will be required in the ensuing chapters. In Section 2 we shall state, without proofs, relevant classical results regarding convex sets and cones. In particular, we concisely review results on extreme points, polyhedrality, support, separation, as well as results concerning dual cones. Several exercises are included; some of these will also be referred to subsequently. In Section 3 we state a required result from real analysis; namely, that the class of Lebesgue measurable functions which map a compact interval into a compact convex set is weakly compact. Section 4 consists of bibliographic notes.

2. CONVEX SETS AND CONES

A set $S \subseteq R^n$ is said to be *convex* if $\alpha x^1 + (1-\alpha)x^2 \in S$ for any points $x^1, x^2 \in S$ and all $\alpha \in [0, 1]$. In other words, the line segment with endpoints x^1 and x^2 is contained in S. Given an arbitrary set $W \subseteq R^n$, we denote the *convex hull* of W by conv(W); this is the smallest convex set containing W. Upon noting that the intersection of an arbitrary number of convex sets is itself convex, it follows that

$$\text{conv}\,(W) = \bigcap \{S \subseteq R^n : W \subseteq S,\ S \text{ convex}\}\ .$$

A vector $x \in R^n$ is said to be a *convex combination* of the vectors $x^i \in R^n$, $i = 1, 2, \ldots, k$, provided that there exist $\alpha_i \geq 0$, $i = 1, 2, \ldots, k$ such that $\Sigma_{i=1}^k \alpha_i = 1$ and $x = \Sigma_{i=1}^k \alpha_i x^i$. Given a set $W \subseteq R^n$, the set of all convex combinations of points in W will be denoted $C(W)$.

(2.1) **Exercise.** For any set W we have $C(W) = \text{conv}\,(W)$.

The following result, known as *Carathéodory's theorem*, strengthens the preceding exercise.

(2.2) **Theorem.** Let W be an arbitrary set in R^n, and let $x \in \text{conv}(W)$. Then there exists a set of points $\{x^i\}_{i=1}^{n+1} \subseteq W$ such that $x \in C(x^1, x^2, \ldots, x^{n+1})$. □

A subset M of R^n is called an *affine set* if $\alpha x^1 + (1-\alpha)x^2 \in M$ for every $x^1 \in M$, $x^2 \in M$ and $\alpha \in R$. Characterizations of affine sets are given in the next exercise.

(2.3) **Exercise.** Let M be a subset of R^n. Then the following are equivalent.

(i) M is an affine set.
(ii) There exists a linear subspace $\mathcal{S} \subseteq R^n$ and a point $p \in M$ such that $M = p + \mathcal{S}$.
(iii) There exists an $m \times n$ matrix B and a vector $b \in R^m$, $m \leq n$, such that $M = \{x \in R^n : Bx = b\}$.

Given a set $W \subseteq R^n$, we denote the *affine hull* of W by aff(W); this is the smallest affine set which contains W. It is readily noted that a convex set S has nonempty interior relative to its affine hull. This relative interior will be denoted ri(S). More generally, we will denote the interior of a set W with respect to an affine set Q by $\text{ri}_Q(W)$. Hence for a convex set S we have $\text{ri}(S) = \text{ri}_{\text{aff}(S)}(S)$. The boundary of W relative to Q is denoted $\text{rb}_Q(W)$. If S is convex, then we shall write $\text{rb}(S) = \text{rb}_{\text{aff}(S)}(S)$. The ordinary interior and boundary of a set W (that is, with respect to R^n) are denoted int W and ∂W, respectively, and the closure of W is denoted cl(W).

(2.4) **Exercise.** Let S be a convex set. Then the following hold.

(i) For any $x^1 \in \text{ri}(S)$ and $x^2 \in \text{cl}(S)$, $\alpha x^1 + (1-\alpha)x^2 \in \text{ri}(S)$ for all $\alpha \in (0, 1]$.
(ii) The sets cl(S) and ri(S) are convex and have the same affine hull as S.

Let $S \subseteq R^n$ be a convex set. A point $x \in S$ is said to be an *extreme point* of S provided that x cannot be expressed as a convex combination of two points in S which are distinct from x. The set of all extreme points of S is denoted by EP(S). A fundamental characterization of compact convex sets in terms of extreme points is given by the following result.

(2.5) **Theorem.** Let $S \subseteq R^n$ be a compact convex set. Then $S = \text{conv}(EP(S))$. □

A set of the form $\mathcal{H} = \mathcal{H}(\alpha, \nu) = \{x \in R^n : \nu^T x = \alpha\}$, where $0 \neq \nu \in R^n$ and $\alpha \in R$ is said to be a *hyperplane*. (At times we will use inner product notation in order to express $\mathcal{H}$ as $\{x \in R^n : \langle \nu, x \rangle = \alpha\}$.) In view of Exercise

2.3, $\mathscr{H}$ is affine and is a translation of an $(n-1)$-dimensional linear subspace. The vector ν is called a *normal* to $\mathscr{H}$, since it is orthogonal to the difference of any two vectors in $\mathscr{H}$. The opposite *closed halfspaces associated with* $\mathscr{H}$ are the sets $\{x \in R^n : \nu^T x \leq \alpha\}$ and $\{x \in R^n : \nu^T x \geq \alpha\}$.

A *face* of a convex set S is a convex subset $S' \subseteq S$ with the property that any line segment $L \subseteq S$ such that $S' \cap \text{ri}(L) \neq \emptyset$ has both endpoints in S'. The empty set as well as any extreme point of S are faces of S. Also, it is readily noted that any extreme point of a face of S is also an extreme point of S.

A set is said to be *polyhedral* if it is expressible as the intersection of a finite family of closed halfspaces. Note that a polyhedral set is closed and convex, but not necessarily bounded. The following result summarizes important facts regarding polyhedrality.

(2.6) **Theorem.** Let S be a convex set. Then the following are equivalent.

(i) S is closed and has a finite number of extreme points.
(ii) S is closed and has a finite number of faces.
(iii) S is polyhedral.

□

In view of Theorems 2.5 and 2.6, a compact convex set S is polyhedral if and only if it is *finitely generated*; that is, S is the convex hull of a finite set of points.

A fundamental separation result for convex sets is given next.

(2.7) **Theorem.** Let S and U be convex subsets of R^n whose relative interiors have empty intersection. Then there exists a hyperplane $\mathscr{H} \subseteq R^n$ such that S and U are contained in opposite closed halfspaces associated with $\mathscr{H}$. What is more, $\mathscr{H}$ may be chosen so that $\mathscr{H} \cap \{\text{rb}(S)\} = \emptyset$ and $\mathscr{H} \cap \{\text{ri}(S)\} = \emptyset$.

□

The hyperplane $\mathscr{H}$ in Theorem 2.7 is said to *separate* the sets S and U. This theorem implies that if S is a convex set and $x \in \partial S$, then there exists a hyperplane $\mathscr{H} = \mathscr{H}(\alpha, \nu)$ containing x (i.e., $\alpha = \nu^T x$) such that S is contained in one of the closed halfspaces associated with $\mathscr{H}$. Then we say that $\mathscr{H}$ *supports* S *at* x, and ν is called a *normal* to S. In case $\mathscr{H} \cap \text{cl}(S) = x$, then we say that $\mathscr{H}$ *strictly supports* S *at* x.

Suppose that S is a convex subset of R^n and $x \in \partial S$. If $0 \neq \nu \in R^n$ satisfies $\langle \nu, y - x \rangle \leq 0$ for all $y \in S$, then ν is called an *outward normal to* S *at* x, and in this case the hyperplane $\mathscr{H}(\nu^T x, \nu)$ supports S at x. The set of all outward unit normals to S at x is denoted by $\mathscr{N}_S(x)$. (Here "unit" means euclidean length equaling unity.) It is readily noted that $\mathscr{N}_S(x)$ is a closed set.

In the next result, a strengthening of the hypotheses of Theorem 2.7 yields correspondingly stronger conclusions.