Lecture Notes in Mathematics

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AMS Subject Classifications (1980): 13A17, 13E05

ISBN 3-540-12722-4 Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 0-387-12722-4 Springer-Verlag New York Heidelberg Berlin Tokyo

Printed in Germany

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2146/3140-543210

Lecture Notes in Mathematics

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ACKNOWLEDGMENTS

Numerous people have participated in the study of asymptotic prime divisors, and I have tried to acknowledge, in the text, a sampling of their contributions. To do so entirely would be impossible, and I hope I have been fair in my selection. Certain people have been particularly helpful to me, as much in stimulating conversations as in specific results. I offer my gratitude to Paul Eakin, Ray Heitmann, Dan Katz and Keith Whittington. My special thanks goes, as it does so often, to Jack Ratliff.

Part of my research was supported by the National Science Foundation, for which I am grateful.

Nita Goldrick typed the manuscript. Her great skill and patience eased a difficult task.

INTRODUCTION

Asymptotic prime divisors represent the interface of two major ideas in the study of commutative Noetherian rings. The first, the concept of prime divisors, is one of the most valued tools in the researcher's arsenal. The second is the fact that in a Noetherian ring, large powers of an ideal are well behaved, as shown by the Artin-Rees Lemma or the Hilbert polynomial.

Although its roots go back further, the recent interest in asymptotic prime divisors began with a question of Ratliff: What happens to $\mathrm{Ass}(R/I^n)$ as n gets large? He was able to answer a related question, showing that if $\overline{I}$ is the integral closure of I, then $\mathrm{Ass}(R/\overline{I^n})$ stabilizes for large n. In a later work, he also showed that $\mathrm{Ass}(R/\overline{I^n}) \subseteq \mathrm{Ass}(R/\overline{I^{n+1}})$. (Earlier, Rees had shown that if $P \in \mathrm{Ass}(R/\overline{I^n})$, some n, then $P \in \mathrm{Ass}(R/\overline{I^m})$ for infinitely many m.) Meanwhile, Brodmann answered the original question, proving that $\mathrm{Ass}(R/I^n)$ also stabilizes for large n. Since then, the topic of asymptotic prime divisors has been growing rapidly, the latest development being the advent of asymptotic sequences, a useful and interesting analogue of R-sequences.

These notes attempt to present the bulk of the present knowledge of asymptotic prime divisors in a reasonably efficient way, to ease the task of those wishing to learn of, or contribute to the subject. Modulo some gnashing of teeth, and rending of garments, it was both educational and satisfying to write them. I hope that reading them is the same.

The first chapter shows that for an ideal I in a Noetherian ring R, $\mathrm{Ass}(R/I^n)$ stabilizes for large n, as does $\mathrm{Ass}(I^{n-1}/I^n)$, the respective stable values of these two sequences are being denoted $A^*(I)$ and $B^*(I)$. Also $B^*(I)$ is characterized as the contraction to R of prime divisors Q of $t^{-1}\mathcal{R}$ with $It \not\subseteq Q$, where $\mathcal{R} = \mathcal{R}[t^{-1}, It]$ is the Rees ring of R with respect to I.

Chapter Two shows that $A^*(I) - B^*(I) \subseteq \mathrm{Ass}\, R$, and that $P \in A^*(I) - B^*(I)$ if and only if there is a $k \geq 1$ such that $P^{(k)}$ is part of a primary decomposition of I^n for all sufficiently large n.

Chapter Three shows that $\mathrm{Ass}(R/\overline{I}) \subseteq \mathrm{Ass}(R/\overline{I^2}) \subseteq \ldots$, and that this sequence eventually stabilizes to a set denoted $\bar{A}^*(I)$. Furthermore, $\bar{A}^*(I) \subseteq A^*(I)$. It also developes several technical results useful for dealing with $\bar{A}^*(I)$, the most important of these being that in a local ring, $P \in \bar{A}^*(I)$ if and only if there are primes $q^* \subseteq p^*$ in the completion R^* such that q^* is minimal, $p^* \cap R = P$ and $p^*/q^* \in \bar{A}^*(IR^* + q^*/q^*)$.

In Chapter Four, it is shown that if R is locally quasi-unmixed, then $P \in \bar{A}^*(I)$ if and only if height $P = \ell(I_P)$, the analytic spread of I_P. Since a complete local domain is locally quasi-unmixed, this result meshes nicely with the one mentioned from Chapter Three.

Chapter Five introduces asymptotic sequences: A sequence $x_1, \ldots, x_n$ such that $(x_1, \ldots, x_n) \neq R$ and for $i = 0, \ldots, n-1$, $x_{i+1} \notin \cup\{P \in \bar{A}^*((x_1, \ldots, x_i))\}$. In a local ring (R,M) it is shown that $x_1, \ldots, x_n$ is an asymptotic sequence if and only if $\mathrm{height}((x_1, \ldots, x_n)R^* + q^*/q^*) = n$ for each minimal prime q^* of the completion. This is then used to show that for a given ideal I in any Noetherian ring, all asymptotic sequences maximal with respect to coming from I have the same length, denoted $\mathrm{gr}^* I$. It is then shown that asymptotic sequences are to locally quasi-unmixed rings as R-sequences are to Cohen-Macaulay rings.

In Chapter Six, the sequence $x_1, \ldots, x_n$ is called an asymptotic sequence over the ideal I if $(I, x_1, \ldots, x_n) \neq R$ and for $i = 0, \ldots, n-1$, $x_{i+1} \notin \cup\{P \in \bar{A}^*((I, x_1, \ldots, x_i))\}$. It is shown that in a local ring, all maximal asymptotic sequences over I have the same length.

Chapter Seven proves that in a local ring, the grade of R/I^n stabilizes for large n, and gives partial results concerning $\mathrm{gr}(R/\overline{I^n})$.

Chapter Eight identifies, with one possible exception, all Noetherian rings for which $A^*(I) = \bar{A}^*(I)$ for all ideals I.

In Chapter Nine, asymptotic prime divisors play a minor role in proving the following unexpected result. Let P be prime in a Noetherian domain. Then there is a chain of ideals $P = I_0 \subset I_1 \subset \ldots \subset I_n$ with the following property: Let Q be a prime containing P, and let j be the largest subscript such that $I_j \subseteq Q$. Then $P \subseteq Q$ satisfies going down if and only if j is even.

In Chapter Ten, we consider a local ring (R,M) and the ideal transform of M, $T(M)$. Previously it was known that the following two statements are equivalent: (a) $T(M)$ is an infinite R-module (b) The completion of R contains a depth 1 prime divisor of zero. Our main result adds two more equivalent conditions: (c) $M \in \bar{A}^*(J)$ for every regular ideal J (d) There is a regular element x with $M \in \bar{A}^*(J)$ for all $J \sim xR$. Here $J \sim I$ if for some n and m, I^n and J^m have the same integral closure. Motivated by statement (d), we then discuss the possibility of defining a strong asymptotic sequence $x_1, \ldots, x_n$ with $(x_1, \ldots, x_n) \neq R$ and for $i = 0, \ldots, n-1$ $x_{i+1} \notin \cup \{P \in \cap \bar{A}^*(J) \mid J \sim (x_1, \ldots, x_i)\}$, in the hope that such a sequence will stand in relation to prime divisors of zero, as asymptotic sequences stand to minimal primes. This program is carried out for $n = 1$ and 2.

Chapter Eleven is aptly titled Miscellaneous. It contains topics (of varying worth) which did not fit elsewhere.

The study of asymptotic prime divisors frequently impinges on that of the structure of the spectrum of a Noetherian ring, often referred to as the study of chain conditions. I have tried to keep to a minimum the amount of knowledge of chain conditions necessary to read these notes. In the Appendix, I list those definitions and basic results (with references for the curious reader) which are referred to in the text.

TABLE OF CONTENTS

CHAPTER I: $A^*(I)$ and $B^*(I)$

DEFINITION. Let I be an ideal in a Noetherian ring R. For $n = 1,2,3,\ldots$, let $A(I,n) = \mathrm{Ass}(R/I^n)$ and let $B(I,n) = \mathrm{Ass}(I^{n-1}/I^n)$.

In [R3], Ratliff asked about the behavior of the sequence $A(I,n)$ (and showed that a related sequence stabilized, see Chapter 3). In [B1], Brodmann showed that both sequences $A(I,n)$ and $B(I,n)$ stabilize for large n, as we now show. Recall that the graded Noetherian ring $T = \Sigma R_n$, $n \geq 0$ is homogeneous if $T = R_0[R_1]$. Our first lemma is well known.

LEMMA 1.1. a) Let $\Sigma_{n\geq 0} R_n$ be a Noetherian homogeneous graded ring. Then there is an ℓ such that for $n \geq \ell$, $(0 : R_1) \cap R_n = 0$.

b) Let I be an ideal in a Noetherian ring. Then there is an ℓ such that for $n \geq \ell$, $(I^{n+1} : I) \cap I^{\ell} = I^n$.

Proof: a) Let $(0 : R_1)$ be generated by homogeneous elements $a_1, \ldots, a_s$. Let $\ell = 1 + \max\{\deg a_i\}$. If $x = \Sigma r_i a_i \in (0 : R_1) \cap R_n$ with $n \geq \ell$, then we may assume the r_i are homogeneous and have $r_i \in R_1 T$. Thus $r_i a_i = 0$, and so $x = 0$.

b) Let $\Sigma R_n = \Sigma I^n/I^{n+1}$ and pick ℓ as above. Say $n \geq \ell$ and let $x \in (I^{n+1} : I) \cap I^{\ell}$. Suppose that $x \notin I^n$. Let $x \in I^k - I^{k+1}$, and note that $\ell \leq k < n$. Since $xI \subseteq I^{n+1} \subseteq I^{k+2}$, with $\bar{x} \in I^k/I^{k+1} = R_k$ we have $0 \neq \bar{x} \in (0 : R_1) \cap R_k$, contradicting part a.

LEMMA 1.2. Let $T = \Sigma R_n$ be a Noetherian graded ring. Let I be a homogeneous ideal and let c be a homogeneous element. Suppose that S is a multiplicatively closed subset of R_0 and that $(I : c) \cap S = \emptyset$. Then there is a homogeneous element d, such that $(I : cd)$ is prime and $(I : cd) \cap S = \emptyset$.

Proof: Among all homogeneous d' with $(I : cd') \cap S = \emptyset$, choose d so that $(I : cd)$ is maximal. It is enough to take homogeneous x and y not in $(I : cd)$ and show $xy \notin (I : cd)$. Suppose, contrarily, that $xy \in (I : cd)$. Then $x \in (I : cdy)$ so that $(I : cdy)$ is strictly larger than $(I : cd)$. Thus there is an $s \in S \cap (I : cdy)$. Now $y \in (I : cds)$, showing that this ideal is strictly larger than $(I : cd)$. Thus there is an $s' \in S \cap (I : cds)$. This gives $ss' \in S \cap (I : cd)$, a contradiction.

PROPOSITION 1.3. [ME] Let $T = \Sigma_{n \geq 0} R_n$ be a Noetherian homogeneous graded ring. Then there exists an m such that $\mathrm{Ass}_{R_0}(R_m) = \mathrm{Ass}_{R_0}(R_n)$ for all $n \geq m$.

Proof: Let $P \in \cup \mathrm{Ass}_{R_0}(R_k)$, $k = 0,1,2,\ldots$. Then for some homogeneous $c \in T$, $P = (0 : c)_{R_0}$. Clearly $P = (0 : c)_T \cap R_0$ and by Lemma 1.2, for some homogeneous $d \in T$ we have $P^* = (0 : cd)$ prime in T and $P^* \cap R_0 = P$. As $\mathrm{Ass}_T(T)$ is finite, we see that $\cup \mathrm{Ass}_{R_0}(R_k)$ is finite.

Now select ℓ as in Lemma 1.1 and say $n \geq \ell$. If $P \in \mathrm{Ass}_{R_0}(R_n)$ write $P = (0 : c)_{R_0}$, $c \in R_n$. As $n \geq \ell$, $P = (0 : cR_1)_{R_0}$. Since $cR_1 \subseteq R_{n+1}$, we have $P \in \mathrm{Ass}_{R_0}(R_{n+1})$. Thus $\mathrm{Ass}_{R_0}(R_n) \subseteq \mathrm{Ass}_{R_0}(R_{n+1})$ for $n \geq \ell$. As we already have $\cup \mathrm{Ass}_{R_0}(R_k)$ finite, the result follows.

COROLLARY 1.4. (Brodmann [B1]) Let I be an ideal in the Noetherian ring R. The sequence $B(I,n)$ stabilizes.

Proof: Apply the proposition to $\Sigma I^{n-1}/I^n$.

COROLLARY 1.5. (Brodmann [B1]) Let I be an ideal in the Noetherian ring R. The sequence $A(I,n)$ stabilizes.

Proof: The exact sequence $0 \to I^n/I^{n+1} \subseteq R/I^{n+1} \to R/I^n \to 0$ shows that $A(I,n+1) \subseteq A(I,n) \cup B(I,n+1)$. For large n, we already have $B(I,n+1) = B(I,n) \subseteq A(I,n)$. Thus $A(I,n+1) \subseteq A(I,n)$, and the result is clear since $A(I,n)$ is finite.

Note that for an ideal I in a Noetherian ring R, $B(I,n) \subseteq A(I,n)$. The following example, due to A. Sathaye, shows that neither sequence is monotone.

EXAMPLE. Let k be a field and n a positive integer. Let $R = k[x, z_1, \ldots, z_{2n}]$ with the restrictions that $xz_{2i-1}^{2i-1} = z_{2i}^{2i}$ for $i = 1,2,\ldots,n$, and $z_j^j z_i = 0$ for

$1 \leq i, j \leq 2n$. Let $I = (z_1, z_2, \ldots, z_{2n}) \subseteq P = (x, z_1, \ldots, z_{2n})$. Then for $1 \leq i \leq 2n$, $P \in B(I,i)$ if i is even, while $P \notin A(I,i)$ if i is odd.

Proof: Since $z_{2i-1}^{2i-1} \notin I^{2i}$ and $Pz_{2i-1}^{2i-1} \subseteq I^{2i}$, we have $P \in B(I,2i)$ for $1 \leq i \leq n$. To see that $P \notin A(I,s)$ for s odd, $1 \leq s \leq 2n$, note that $P \notin A(I,1)$ since I is prime. Now for $1 \leq q \leq 2n$, the residues of the set $T_q = \{z_q^q\} \cup \{z_2^{u_2} \ldots z_{2n}^{u_{2n}} | u_2 + \ldots + u_{2n} = q,\ 0 \leq u_i < i\}$ form a generating set for I^q/I^{q+1} over $k[x]$. If q is even, there are no relations, and T_q gives a free basis. If q is odd, there is the unique relation $xz_q^q \in I^{q+1}$. Suppose $P \in A(I,s)$. Then for some $w \notin I^s$, $Pw \subseteq I^s$. Consider r such that $w \in I^r - I^{r+1}$. By the previous remarks, $xw \in I^{r+1}$ shows that r is odd. Furthermore, it can be seen that $xw \notin I^{r+2}$. Thus $s = r + 1$ and so s is even.

DEFINITION. For I an ideal in a Noetherian ring, the eventual constant values of the sequences $A(I,n)$ and $B(I,n)$ will be denoted $A^*(I)$ and $B^*(I)$, respectively.

The fact that $A^*(I)$ and $B^*(I)$ behave well under localization is straightforward, and yet we will use it so often that we state it formally.

LEMMA 1.6. Let $I \subseteq P$ be ideals in a Noetherian ring, with P prime. Then $P \in A^*(I)$ (respectively $P \in B^*(I)$) if and only if $P_S \in A^*(I_S)$ (respectively $P_S \in B^*(I_S)$), for any multiplicatively closed set S disjoint from P.

The next result will lead to some interesting applications of asymptotic prime divisors. As this result will be used again when discussing the integral closure of an ideal (Chapter 3), we give it here in full generality.

If J is an ideal of R, we will use $\overline{J}$ to denote the integral closure of J. Thus $\overline{J} = \{x \in R | x$ satisfies a polynomial of form $X^n + j_1X^{n-1} + \ldots + j_n = 0$, with $j_i \in J^i\}$. Recall that $\overline{R}$ is the integral closure of R.

PROPOSITION 1.7. [M3] Let P be a prime ideal in a Noetherian domain R. There is an integer $n \geq 1$ with the following property: If I is an ideal of R with

$I \subseteq \overline{P^n}$, and if there exists an integral extension domain T of R and a $Q \in \operatorname{spec} T$ with $Q \cap R = P$ and Q minimal over IT, then $P \in \operatorname{Ass}(R/I)$.

Proof: Let $P_1, \ldots, P_m$ be all of the primes of $\overline{R}$ which lie over P. Select $u_i \in P_i - \cup P_j$, $j \neq i$, and let $S = R[u_1, \ldots, u_m]$. Notice that P_i is the unique prime of $\overline{R}$ lying over $p_i = P_i \cap S$. Let (V_i, N_i) be a D.V.R. overring of S with $N_i \cap S = p_i$. Since S is a finitely generated R-module, we can choose $b \in R$ with $bS \subseteq R$. Pick n sufficiently large that $b \notin N_i^n$, $i = 1,2,\ldots,m$.

Suppose that $I \subseteq \overline{P^n}$ and that T is an integral extension domain of R containing a prime Q with $Q \cap R = P$ and Q minimal over IT. We first reduce to the case that $T = S$. Clearly we may assume $T = \overline{T}$, and by going down we may replace $\overline{T}$ by $\overline{R}$. Finally since P_i is the only prime of $\overline{R}$ lying over p_i, $i = 1,2,\ldots,m$, by going up we replace $\overline{R}$ by S.

We now have $T = S$, and of course $Q = p_i$ for some $i = 1,2,\ldots,m$. We localize making P maximal in R. Since p_i is minimal over IS, there is an integer $k \geq 1$ and an $s \in S - p_i$ with $sp_i^k \subseteq IT$. Using $bS \subseteq R$, we have $bsP^k \subseteq bsp_i^k \subseteq bIT \subseteq I$. Furthermore, we claim $bs \notin \overline{I}$. If $bs \in \overline{I} \subseteq \overline{P^n} \subseteq \overline{p_i^n} \subseteq \overline{N_i^n} = N_i^n$, then since $s \notin p_i$ implies s is a unit of V_i, we have $b \in N_i^n$, contradicting our choice of n. Thus $bs \notin I$ but $bsP^k \subseteq I$, showing that P^k consists of zero divisors modulo I. As P was maximal, $P \in \operatorname{Ass}(R/I)$.

COROLLARY 1.8. Let I be an ideal in a Noetherian domain R and let T be an integral extension domain of R. If Q is prime in T and minimal over IT, then $Q \cap R \in A^*(I)$.

Proof: Let $P = Q \cap R$, and choose n as in the proposition. Then $P \in A(I,m)$ for $m \geq n$.

The following fact about the integral closure of a Noetherian domain appears to depend upon knowledge of asymptotic prime divisors.

PROPOSITION 1.9. Let R be a Noetherian domain. Let $J = (b_1, \ldots, b_m)$ be a finitely generated ideal of the integral closure $\overline{R}$. Then the number of primes of $\overline{R}$ minimal over J is finite.

Proof: Let $S = R[b_1, \ldots, b_m]$ and let $I = (b_1, \ldots, b_m)S$. Thus S is Noetherian and $I\overline{R} = J$. If $Q \in \operatorname{spec} \overline{R}$ and Q is minimal over J, then by Corollary 1.8, $Q \cap S \in A^*(I)$. Since $A^*(I)$ is finite and since only finitely many primes of $\overline{R}$ lie over a given prime on S, we are done.

We generalize [N, 33.11].

PROPOSITION 1.10. Let $R \subseteq T$ be an integral extension of domains with R Noetherian. Let Q be a height n prime of T and let $P = Q \cap R$. Then grade $P \leq n$. If grade $P = n$, then for any R-sequence $a_1, \ldots, a_n$ coming from P, P is a prime divisor of $(a_1, \ldots, a_n)$.

Proof: We induct on n. For $n = 1$, pick $a \neq 0$ in P. Since height $Q = 1$, Q is minimal over aT. By Corollary 1.8, for sufficiently large k, P is a prime divisor of a^kR. It is not difficult to now see that P is also a prime divisor of aR.

For $n > 1$, suppose grade $P \geq n$ and let $a_1, \ldots, a_n$ be an R-sequence coming from P. We claim height$(a_1, \ldots, a_n)T = n$. If not, say $q \in \operatorname{spec} T$, height $q < n$ and $(a_1, \ldots, a_n)T \subseteq q$. By induction, grade $q \cap R \leq$ height $q < n$, contradicting that $a_1, \ldots, a_n$ is an R-sequence in $q \cap R$. Thus the claim is true, and so Q is minimal over $(a_1, \ldots, a_n)T$. By Corollary 1.8, for large k we have P a prime divisor of $(a_1, \ldots, a_n)^k$ in R. As $a_1, \ldots, a_n$ is an R-sequence, P is also a prime divisor of $(a_1, \ldots, a_n)$ by [K1, Section 3-1, Exercise 13].

In Chapter 5 we strengthen Proposition 1.10, replacing "height Q" by "little height Q".

The next three propositions give easy circumstances under which a prime must be in $A^*(I)$.

PROPOSITION 1.11. Let I be an ideal in a Noetherian ring, and let the prime P be minimal over I. Then $P \in A^*(I)$. Also $P \in B^*(I)$ if and only if height $P > 0$.

Proof: Since P is minimal over I^n for all n, $P \in A(I,n)$, and so $P \in A^*(I)$. For the second statement, localize at P, so that I is P-primary. Now height $P > 0$ if and only if I is not nilpotent. If I is nilpotent, clearly $P \notin B^*(I)$. If I is not nilpotent, then for all n I^n/I^{n+1} is a nonzero module (by Nakayama's Lemma) which must have at least one prime divisor. However P is the only possibility. Thus $P \in B(I,n)$ for all n.

PROPOSITION 1.12. Let $I \subseteq P$ with P a prime divisor of 0 in a Noetherian ring. Then $P \in A^*(I)$.

Proof: Localize at P and then write $P = (0 : c)$. For n large enough that $c \notin I^n$, clearly $P = (I^n : c)$.

Our next proposition generalizes Proposition 1.11. The lemma is due to Ratliff.

LEMMA 1.13. Let $Q \subset P$ be primes of the Noetherian ring R such that Q is a prime divisor of 0. Then there is an integer $n > 0$ such that for any ideal J of R with $J \subseteq P^n$ and P minimal over $Q + J$, we have $P \in \mathrm{Ass}(R/J)$.

Proof: Localize at P. Let $q_1 \cap \ldots \cap q_r$ be a primary decomposition of 0 with q_1 primary to Q. Choose $0 \neq x \in q_2 \cap \ldots \cap q_r$, and pick n such that $x \notin P^n$. Suppose that $p \in \mathrm{Ass}(R/J)$ and $p \neq P$. Since P is minimal over $Q + J$, we have $Q \not\subseteq p$. Thus in R_p, $0 = (q_2)_p \cap \ldots \cap (q_r)_p$ so that $xR_p = 0$. This shows that x is in every p-primary ideal. However, $J \subseteq P^n$ shows that $x \notin J$. Thus $P \in \mathrm{Ass}(R/J)$, using primary decomposition.

PROPOSITION 1.14. Let I, P, Q be ideals in a Noetherian ring with Q a prime divisor of 0, and P a prime minimal over $Q + I$. Then $P \in A^*(I)$.

Proof: With n as in Lemma 1.13, $P \in A(I,m)$ for all $m \geq n$.

Later (Proposition 2.5) we will strengthen Proposition 1.14 to say that if in addition $P \neq Q$, then $P \in B^*(I)$.

We give a characterization of $B^*(I)$ in terms of the Rees ring of R with respect to I, that is, the ring $\mathcal{R} = R[t^{-1}, It]$ with t an indeterminate.

PROPOSITION 1.15. Let I be an ideal in the Noetherian ring R, and let $\mathcal{R} = R[t^{-1}, It]$ be the Rees ring of R with respect to I. Then $P \in B^*(I)$ if and only if there is a prime divisor Q of $t^{-1}\mathcal{R}$ such that $It \not\subseteq Q$ and $Q \cap R = P$.

Proof: Let $P \in B^*(I)$. Consider ℓ as in Lemma 1.1b, and choose $n > \ell$ with $P \in \mathrm{Ass}(I^n/I^{n+1})$. Write $P = (I^{n+1} : c)$ with $c \in I^n$. Since $ct^n \in \mathcal{R}$, note that $(t^{-1}\mathcal{R} : ct^n) \cap R = (I^{n+1} : c) = P$. By Lemma 1.2 there is a $dt^m \in \mathcal{R}$ such that $Q = (t^{-1}\mathcal{R} : dct^{n+m})$ is prime in $\mathcal{R}$ and $Q \cap R = P$. We must show that $It \not\subseteq Q$. Since Q is a proper ideal, $dct^{n+m} = (dt^m)(ct^n) \notin t^{-1}\mathcal{R}$. Thus $m \geq 0$ and $dc \notin I^{n+m+1}$. By Lemma 1.1, $(I^{n+m+2} : I) \cap I^{\ell} = I^{n+m+1}$, and since $c \in I^n$ we must have $dc \notin (I^{n+m+2} : I)$. Therefore $It \not\subseteq (t^{-1}\mathcal{R} : dct^{n+m}) = Q$ as desired.

Conversely, suppose that $Q = (t^{-1}\mathcal{R} : gt^k)$ with $g \in I^k$, that $Q \cap R = P$, and that $It \not\subseteq Q$. Pick $ht \in It - Q$. Clearly $Q = (t^{-1}\mathcal{R} : gh^m t^{k+m})$ for all $m > 0$. Thus $P = (I^{k+m+1} : gh^m)$. Since $gh^m \in I^{k+m}$ and m is arbitrary, we have $P \in B^*(I)$.

We close the chapter with a question. We have seen that the sequence $A(I,1), A(I,2), \ldots$ is not increasing. Ratliff asks whether $A(I,1) \cap A^*(I)$, $A(I,2) \cap A^*(I), \ldots$ is increasing?

CHAPTER II: $A^*(I) - B^*(I)$

In this chapter, we study primes contained in $A^*(I)$ but not $B^*(I)$, our main result being that such primes must be prime divisors of zero.

LEMMA 2.1. Let I be an ideal in a Noetherian ring R and (by Lemma 1.1) suppose for $n > \ell$ we have $(I^n : I) \cap I^\ell = I^{n-1}$. If P is prime in R and if $P = (I^n : c)$ with $n > \ell$ and $c \in I^\ell$, then $P \in B^*(I)$.

Proof: Since $cI \subseteq cP \subseteq I^n$, we have $c \in (I^n : I) \cap I^\ell = I^{n-1}$. For $j \geq 0$, clearly $P \subseteq (I^{n+j} : cI^j)$. Conversly, if $r \in (I^{n+j} : cI^j)$ then $rcI^{j-1} \subseteq (I^{n+j} : I) \cap I^\ell = I^{n+j-1}$, so that $r \in (I^{n+j-1} : cI^{j-1})$. Iterating, we find $r \in (I^n : c) = P$. Thus $P = (I^{n+j} : cI^j)$ for $j = 1,2,\ldots$. Now we already have $c \in I^{n-1}$, so $cI^j \subseteq I^{n+j-1}$. Thus $P \in B(I, n+j)$ for $j = 1,2,\ldots$.

PROPOSITION 2.2 [ME] Let I be an ideal in a Noetherian ring R. If $P \in A^*(I) - B^*(I)$ then P is a prime divisor of zero.

Proof: We may localize at P. Since $P \in A^*(I)$, for all large n we have an $x_n \in R$ with $P = (I^n : x_n)$, and by Lemma 2.1 we have $x_n \notin I^\ell$. To show that P is a prime divisor of zero, it is sufficient to show this in the case that (R,P) is complete, which we now assume. Let $V = (I^\ell : P)/I^\ell$ and for $n > \ell$ let $V_n = [(I^n : P) + I^\ell]/I^\ell$. Now $PV = 0$, so V is a finite dimensional vector space over R/P. Clearly x_n taken modulo I^ℓ is a nonzero element in the subspace V_n. Since $V_{n+1} \subseteq V_n$, we see that $\cap V_n \neq 0$, by finite dimensionality. Let $\bar{\lambda} \neq 0$ be in this intersection, and let $\lambda \in (I^\ell : P) - I^\ell$ be a preimage. Since $\bar{\lambda} \in V_n$ write $\lambda = d_n + i_n$ with $d_n \in (I^n : P)$ and $i_n \in I^\ell$. For $m \geq n$ we have $I(i_n - i_m) \subseteq P(i_n - i_m) = P(d_n - d_m) \subseteq I^n$. Thus $i_n - i_m \in (I^n : I) \cap I^\ell = I^{n-1} \subseteq P^{n-1}$, showing that the sequences $\{i_n\}$ and $\{d_n\}$ are Cauchy sequences. Let $i_n \to i$ and $d_n \to d$. Since $i \in I^\ell$ and $\lambda \notin I^\ell$, $d \neq 0$. Finally, since $d_n P \subseteq I^n \subseteq P^n$, $dP \subseteq \cap P^n = 0$, concluding the proof.