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J. Noguchi T. Ohsawa (Eds.)

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Preface

The 25th Taniguchi International Symposium, Division of Mathematics, titled with "Prospects in Complex Geometry" was held at Kyuzeso, Katata, July 31 through August 5, 1989, and an international conference with the same title was held successively at Research Institute for Mathematical Sciences (RIMS), Kyoto University, Kyoto, August 7 through August 9. The present volume consists of papers based on talks given at the two meetings. The central subject was complex structure and the emphasis was put on geometric aspects. The topics of the papers range therefore over various materials from complex function theory in one variable to differential geometry and algebraic geometry; e.g., the Teichmüller theory, the deformation theory of special complex manifolds, the moduli theory of holomorphic and harmonic mappings, and the cohomology theory on algebraic varieties.

The International Symposium at Kyuzeso, Katata, was fully and generously supported by the Taniguchi Foundation. The international conference at RIMS, Kyoto University was jointly supported by RIMS and the Taniguchi Foundation. The organizers wish to express their deepest gratitude to Mr. Taniguchi and the Taniguchi Foundation for supporting the two meetings and their warm hospitality, and to RIMS for supporting the second meeting. They are also very grateful to Professor S. Murakami for serving the Taniguchi International Symposium as coordinator, and last but not least to all the participants, speakers and the contributors of this volume.

All papers of this volume are in final form and no similar version will be published elsewhere.

July, 1990

Organizers
J. Noguchi (Tokyo Inst. of Tech.)
T. Ohsawa (RIMS, Kyoto Univ.)

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HYPERKÄHLER STRUCTURE ON THE MODULI SPACE OF FLAT BUNDLES

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Introduction

Let X be a compact Riemann surface. Then by the results of Hitchin [H1] and Donaldson [D] there exists a natural bijective correspondence between the set $\mathfrak{M}$ of isomorphism classes of stable Higgs bundles on X with vanishing chern classes and the set $\mathfrak{N}$ of equivalence classes of irreducible complex representations of the fundamental group of X. It has turned out that such a bijective correspondence still exists for any compact Kähler manifold of higher dimension by the works of Simpson [S1,S2] and Corlette [C] (cf. also [JY]). Moreover, Simpson [S3] has proved that $\mathfrak{M}$ has a natural structure of a quasi-projective scheme when X is projective, and that with respect to this and the well-known similar structure on $\mathfrak{N}$ the above correspondence is homeomorphic.

On the other hand, in the case of a compact Riemann surface Hitchin [H1] had shown that $\mathfrak{M}$ has moreover a natural structure of a hyperkähler manifold such that if $f: Z \to \mathbb{P}$ is the corresponding Calabi family (cf.(2.2) below), the fibers Z_t are complex analytically isomorphic to $\mathfrak{N}$ for all $t \neq 0, \infty$, and Z_0 and Z_∞ are isomorphic to $\mathfrak{M}$ and its complex conjugate respectively, where $\mathbb{P}$ is the complex projective line. In fact, it was shown that the natural $\mathbb{C}^*$-action on $U := \mathbb{P}-\{0,\infty\}$ lifts to a holomorphic $\mathbb{C}^*$-action on Z. The main purpose of this paper is then to show that the same result is also true in the higher dimensional cases at least when we consider only the set of non-singular points. (See Theorems (1.4.1) and (8.3.1) below.) In fact, the basic idea for the proof remains just the same as in [H1] in view of the above mentioned results of Corlette and Simpson.

In Section 1 we shall formulate and state the results in the case of vector bundles though we work more generally in the framework of principal bundles, following the formulation by Ramanathan (cf.[R][RS]); see [S2] for another approach to the principal case. In Sections 2 and 3 the basic definitions and results concerning hyperkähler manifolds and hyperkähler moment maps are summarized and the method of hyperkähler quotients will be explained; here the emphasis is laid on the case where the manifold admits a special S^1-action. In Section 4 we discuss the simplest but basic example of a hyperkähler vector space which is already in [H1]. In relation with the description there, in (4.7) we shall see how the hyperkähler structure on the moduli space looks like in the simplest case of line bundles, a glance at which could be helpful in understanding the general situation.

In the one dimensional case the hyperkähler quotient construction leads directly to the moduli space in question, while in the higher dimensional case the same construction leads only to an infinite dimensional hyperkähler manifold. The description of the latter is the purpose of Sections 5 and 6. The result is stated in Theorem (6.6.1) (cf. also Theorem (1.5.2)). The finite dimensional moduli space in question turns out to be a hyperkähler subspace of the above infinite dimensional manifold; this will be verified in Sections 7 and 8 using the identification $\mathfrak{M} = \mathfrak{N}$ mentioned above. In Section 7 we discuss the non-flat case also. We prove in Section 9 the existence of the moduli space of stable principal Higgs bundles in general as an analytic space along the line of Ramanathan [R]. In the Appendix we have included the proofs of certain results of Hithcin [H1] and Simpson [S1] for the convenience of the reader.

This article was written without knowing the details of the content of [S2]; as a result some part of the paper (e.g. the first part of Section 9) could have been directly quoted from [S2], but we leave it in the original form for lack of time.

During the conference Tsuji pointed out that in the projective case the result could also be obtained by reducing to the one dimensional case by considering the general curve section of X. In fact, this would be possible by using the Higgs version of the theorem of Mehta-Ramanathan due to Simpson [S2], though we hope that our direct construction has its own interest.

1. Statement of Results

(1.1) Let X be a connected compact Kähler manifold with a fixed Kähler metric g. A *Higgs (vector) bundle* over X is a pair (E,θ) consisting of a holomorphic vector bundle E on X and an End E-valued holomorphic 1-form θ on X such that θ satisfies the integrability condition $[\theta,\theta] = 0$. A Higgs bundle (E,θ) is called *stable if* we have $\mu(\mathcal{F}) < \mu(E)$ for any θ-invariant torsion free coherent analytic subsheaf $\mathcal{F}$ of $\mathcal{O}(E)$. Here for any torsion free coherent analytic sheaf $\mathcal{G}$ on X, $\mu(\mathcal{G})$ denotes the rational number $\mu(\mathcal{G}) = \deg \mathcal{G} / \operatorname{rank} \mathcal{G}$ where $\deg \mathcal{G}$ is the degree of $\mathcal{G}$ with respect to the Kähler class γ of g (cf.(7.4.2) below).

Let (E,θ) be a Higgs bundle. Let h be a hermitian metric on E and D_h the associated hermitian connection with the curvature form F_h. Using the metric h we form the conjugate θ^* of θ which is an End E-valued (0,1)-form (cf.(7.1)). Set $\psi = \theta + \theta^*$. Then define another (affine) connection $D = D(E,\theta,h)$ on E by $D = D_h + \psi$. We quote a special case of a basic result of Simpson [S1][S2] on Higgs bundles as a lemma.

(1.1.1) **Lemma.** Let (E,θ) be a Higgs bundle of rank r with the vanishing first and second chern classes. Then (E,θ) is stable if and only if there exists a hermitian metric h on E such that the associated connection D is flat and irreducible. Moreover, in this case such a metric is unique.

(1.2) The above objects are also related to harmonic metrics. In general let $V \to X$ be a C^∞ vector bundle on X and D a connection on V. Given a hermitian metric h on V we may decompose D uniquely into: $D = D_h + \psi$, where D_h is a metric connection and ψ is a C^∞ 1-form with values in self-adjoint endomorphisms of V with respect to h. In this notation we call h *harmonic* (with respect to D) if $D_h\psi = D_h^*\psi = 0$, where D_h^* denotes the formal adjoint of D_h. Note that the first condition is automatic if D is flat (cf.(7.1.1)). The next result is due also to Simpson [S2]. (See (7.7.1) below.)

(1.2.1) **Lemma.** Let (E,θ) be a Higgs bundle and h a hermitian metric such that $D = D(E,\theta,h)$ is flat. Then h is harmonic with respect to D on $V = E$. Conversely, if V is a C^∞ vector bundle with a flat connection D, and a harmonic metric h with respect to D, then the following hold true: 1) the curvature F_h of D_h is of type (1,1) (so that the (0,1)-part D''_h of D_h defines a holomorphic structure on V), 2) (1,0)-part θ of ψ is holomorphic with respect to this holomorphic structure, and 3) θ is integrable; $[\theta,\theta] = 0$.

We must also recall another basic result due to Corlette [C]:

(1.2.2) **Lemma.** Let V be a C^∞ vector bundle and D a flat and irreducible connection on V. Then there exists a unique harmonic metric h with respect to D.

(1.3) We next consider the moduli space of the objects considered above. Let π be the fundamental group X. Fix a positive integer r and set $G = GL(r,\mathbb{C})$. Then the following is well-known (cf. [JM]):

(1.3.1) **Lemma**. The set $\mathfrak{R}$ of equivalence classes of irreducible representations of π into G has a natural structure of a quasi-projective scheme, and hence of a complex analytic space.

Proof. Let R be the set of all the representations $\rho: \pi \to G$. Then R has the natural structure of a complex affine algebraic scheme such that the action of G on R by conjugations is algebraic. An element $\rho \in R$ is irreducible if and only if the image of ρ is not contained in any parabolic subgroup of G. Hence by [JM;Th.1] ρ is irreducible if and only if the G-orbit of ρ is closed in R and the identity component of the stabilizer of ρ coincides with the center of G. A general result of geometric invariant theory [M; Chap.1,§2] yields the desired result.

On the other hand, it is not difficult to show the following:

(1.3.2) **Proposition**. The coarse moduli space of stable Higgs bundles of a fixed rank r exists as a (Hausdorff) complex analytic space.

See Section 9 for the more detail. We are interested in those connected components of the above moduli space corresponding to Higgs bundles with $c_1 = c_2 = 0$. Let $\mathfrak{M}$ be the union of all such components. From the preceeding three lemmas we get:

(1.3.3) **Lemma**. There is a natural bijection between the two moduli spaces $\mathfrak{M}$ and $\mathfrak{R}$.

However, the complex structure of $\mathfrak{R}$ depends only on the underlying topological space of X, while that of $\mathfrak{M}$ depends on the complex structure of X.

(1.4) The main purpose of this paper is to show that these two complex structures $\mathfrak{M}$ and $\mathfrak{N}$ (on the same set) appears as generic and special members of a Calabi family of a certain hyperkähler space (as far as the nonsingular points are concerned for the moment). Denote by $\bar{\mathfrak{M}}$ the complex space which is complex conjugate to $\mathfrak{M}$, and by $\mathbb{P}$ the complex projective line. Let $\mathfrak{M}_0$ and $\mathfrak{N}_0$ be the set of nonsingular points of the underlying reduced subspace of $\mathfrak{M}$ and $\mathfrak{N}$ respectively.

(1.4.1) **Theorem.** The notations and assumptions being as above, there exists a hyperkähler manifold M with a special S^1 action such that in the associated Calabi family $\{Y_t\}_{t\in\mathbb{P}}$ the fiber Y_0 (resp. Y_∞) over (resp. ∞) is isomorphic to $\mathfrak{M}_0$ (resp. $\bar{\mathfrak{M}}_0$) and the other members Y_t are all isomorphic to $\mathfrak{N}_0$.

The result is due to Hitchin (cf.[H1;§§5,6,9]) in case dim X = 1. For the definitions of the terminologies used above see (2.2)-(2.4) below.

(1.5) The proof proceeds roughly as follows. Let $V \to X$ be a fixed C^∞ complex vector bundle of rank, say r. Fix an integer $k \geq \dim X + 2$. Let $\mathcal{A} = \mathcal{A}_k$ be the space of connections on V which are of Sobolev class H_k. $\mathcal{A}$ is a complex affine space with traslation group $A^1_k = A^1_k(X,\mathrm{End}\ E)$ of End E-valued 1-forms of class H_k. The group $\mathcal{G} = \mathcal{G}_{k+1}$ of complex gauge transformations of V (or of the associated principal bundle) of class H_{k+1} acts on $\mathcal{A}$ by complex affine transformations. A connection D of $\mathcal{A}$ with curvature F is said to be *Einstein*, if

(1.5.1) $\sqrt{-1}\Lambda F = \lambda I, \qquad \lambda = 2\pi n\ \mu(E)/\int_X \omega^n$,

where ω is the Kähler form associated to the given Kähler metric g and Λ is the trace operator with respect to g (cf.[W;p.21]). Then we consider the subset $\mathcal{E} = \mathcal{E}_k$ of $\mathcal{A}$ consisting of irreducible Einstein connections D of class H_k which admits a weakly harmonic metric, where we call a metric h on V *weakly harmonic* if $D_K^* \psi = 0$ in the previous notation. The action of $\mathscr{G}$ on $\mathscr{A}$ preserves $\mathscr{E}$ and we form the quotients $\mathfrak{C} = \mathscr{E}/\mathscr{G}$. The main point then is to prove the following:

(1.5.2) **Theorem.** $\mathfrak{C}$ has a natural structure of a Kähler symplectic Hilbert manifold. Moreover, there exists a hyperkähler Hilbert manifold $\mathfrak{F}$ with a special S^1 action such that in the associated Calabi family $\{\mathfrak{Z}_t\}_{t \in P}$, $\mathfrak{Z}_j$ is naturally identified with $\mathfrak{C}$ as a Kähler symplectic manifold.

Here, a Kähler symplectic manifold is a Kähler manifold with a fixed nondegenerate holomorphic 2-form which is parallel with respect to the Levi-Civita connection. $\mathfrak{F}$ is obtained as a hyperkähler quotient associated to a certain hyperkähler moment map in the sense of [HKLR]. (The proof will be given at the end of Section 6.) When dim X = 1, $\mathfrak{F}$ is of finite dimension and already gives the hyperkähler manifold of Theorem (1.4.1) (cf. Hithcin [H1]). In higher dimensional case it is necessarily of infinite dimension and we check that $\mathfrak{N}_0$ is naturally a hyperkähler submanifold of $\mathfrak{F}$ above.

§2. Hyperkähler Moment Map

(2.1) Let $\mathbb{H}$ be the algebra of real quaternions with $\mathbb{R}$-basis denoted by 1, *i*, *j*, *k* as usual. Set

$$C = \{q \in \mathbb{H};\ q^2 = -1\}.$$

Then we have a natural identification $C = \mathbb{H}^*/\mathbb{C}^* = \mathbb{P}$, where $\mathbb{H}^* := \mathbb{H} - 0$ acts on C by inner automorphisms, $\mathbb{C}^*$ is identified with the stabilizer $\mathbb{H}_i^*$ of *i*, and $\mathbb{P}$ denotes the complex projective line. Let

$Sp(1)$ be the group of unit quaternions and $\mathfrak{s} = \mathfrak{sp}(1)$ its Lie algebra, identified with the oriented space of pure quaternions, where i, j and k form an oriented basis. For any element $q \in C \subset \mathfrak{s}$ take an element r and s of C such that q, r and s form an oriented orthonormal basis of $\mathfrak{s}$. (Denote by A_q the set of pairs (r,s) of such elements in $\mathfrak{s}$.) Then the complex line

$$\ell_q := \mathbb{C}(r + \sqrt{-1}s)$$

in the complexification $\mathfrak{s}^{\mathbb{C}}$ is independent of the choices of $(r,s) \in A_q$ and $L := \bigcup_{q \in C} \ell_q$ becomes a holomorphic line bundle of degree -2 on $C = \mathbb{P}$ as can be checked easily (cf. e.g.[F3;p.111,Lemma 1.3]). If we identify $\mathfrak{s}^{\mathbb{C}} \simeq \mathfrak{sl}(2,\mathbb{C})$ with its dual by using the Killing form, any element of $\mathfrak{s}^{\mathbb{C}}$ defines a linear form on each fiber ℓ_q of L depending holomorphically on q; in this way we may identify $\mathfrak{s}^{\mathbb{C}}$ further with the space $\Gamma(C,L^*)$ of holomorphic sections of the dual bundle L^*.

(2.2) We recall the notions of a hyperkähler manifold and the associated Calabi family (cf.[HKLR][F3;p.125]). A *hyperkähler manifold* is a Riemannian manifold (M,g) endowed with almost complex structures I and J such that $IJ = -JI$ and that $(M,I;g)$ and $(M,J;g)$ are Kähler manifolds. In this case we denote the hyperkähler manifold by a quadruple $M = (M,g;I,J)$. Each element

(2.2.1) $$q = ai + bj + ck \in C, \quad a^2 + b^2 + c^2 = 1,$$

defines an almost complex structure $J_q = aI + bJ + cIJ$ on M, which induces a family of Kähler structures (M_q,g) parametrized by $C = \mathbb{P}$, where $M_q = (M,J_q)$ is the underlying complex manifold. The corresponding Kähler form ω_q on M_q is then given by

$$\omega_q(x,y) = g(J_q x,y).$$

Furthermore, for any $u := e(r+\sqrt{-1}s) \in \ell_q$ with a unique $(r,s) \in A_q$ and a unique positive number e,

$$\varphi_u := e(\omega_r + \sqrt{-1}\omega_s)$$

is a holomorphic symplectic 2-form on M_q.

Moreover, the complex manifolds M_q fit well into a holomorphic family in the following sense: There is a unique complex structure $\mathcal{J}$ on $M \times C$ which restricts to J_q on $M = M \times q$ and to the standard complex structure on each $x \times C = \mathbb{P}$. Let $Z = (M \times C, \mathcal{J})$ be the resulting complex manifold. By construction *a*) the natural projection $f: Z \to C$ is holomorphic with fiber over q identical to M_q, and *b*) $x \times C$ is a complex submanifold of Z for any $x \in M$. We call this holomorphic family

$$\{(M_q, g)\}_{q \in C} \quad \text{or} \quad f: Z \to C$$

of Kähler manifolds the *Calabi family* associated to the hyperkähler manifold M. We also note that the association $u (\in \ell_q) \to \varphi_u$ allows us to identify the line bundle f^*L as a subbundle of the bundle $\Lambda^2 T^*_{Z/C}$ of relative holomorphic 2-froms on Z over C.

The notions of a hyperkähler manifold and a Calabi family as well as all those introduced in what follows can naturally be extended to the category of V-manifolds. We leave it to the reader to take care of the relevant details though we shall use freely the corresponding terminology for V-manifolds.

(2.3) We write $i = 0$ and $-i = \infty$ on $C = \mathbb{P}$, and accordingly, $M_0 = M_i$ and $M_\infty = M_{-i}$. The circle group $S^1 \simeq \mathbb{H}^*_i \cap Sp(1) \subset \mathbb{H}^*$ acts naturally on C, and its complexification $\mathbb{C}^*$ in $SL(2,\mathbb{C})$ with respect to the inclusion $Sp(1) \subset SL(2,\mathbb{C})$ acts transitively on $U := C - \{0,\infty\}$. Then a hyperkähler manifold M as above is said to admit a *special* S^1-*action* if there exists a C^∞ S^1-action on the Riemannian manifold (M,g) with the following properties: 1) the above $\mathbb{C}^*$-action on C lifts to a holomorphic $\mathbb{C}^*$-action on Z via f such that it induces the product S^1-action on $Z = M \times C$, and 2) for any $q \in C$, $t \in S^1$ induces a Kähler isometry $(M_q, g) \to (M_{t(q)}, g)$. We speak also of a *special* $\mathbb{C}^*$-*action* on Z in this case. Note that in this case all the (M_q, g), $q \in U$, are isomorphic as Kähler manifolds; hence M cannot be compact; otherwise by Prop. 13 of "A. Fujiki, Publ. RIMS, Kyoto Univ., 20 (1984)"

M_q would all be isomorphic, which is absurd as is well-known.

We get a holomorphic involution on Z induced by $i \in S^1$; in particular it induces an anti-holomorphic involution σ_j on M_j (since i sends j into $-j$ on C and $J_j = -J_{-j}$), and a holomorphic involution σ_0 on M_0. The respective fixed point sets F_0 and F on M_0 and M_j are of course identical with respect to the natural identification $M_0 = M_j = M$. However, F_0 is a complex submanifold of M_0, while F is a 'real part' of M_j in a neighborhood of F.

(2.4) Let $M = (M,g;I,J)$ be a hyperkähler manifold as above. Let $\dim M = 4n$. Then the structure group of the tangent bundle of M is naturally reduced to the unitary symplectic group $Sp(n)$, and hence the tangent bundle admits a natural quaternion inner product $\langle\ ,\ \rangle$ whose real part is precisely the given Riemannian metric g. The $\mathfrak{s}$-part $\langle\ ,\ \rangle_{\mathfrak{s}}$ of $\langle\ ,\ \rangle$ with respect to the decomposition $\mathbb{H} = \mathbb{R} + \mathfrak{s}$ is an $\mathfrak{s}$-valued 2-form on M, which we shall denote by $\omega = \omega_{HK}$ and call the *hyperkähler form* on M. We may write $\omega = i \times \omega_i + j \times \omega_j + k \times \omega_k$.

Let G be a connected complex linear reductive Lie group acting holomorphically on the complex manifold (M,J) and K a maximal compact subgroup of G whose induced action on M preserves the hyperkähler structure, i.e. preserves also I and g. We then recall the notion of the hyperkähler moment map associated to the action of K on M as described in [HKLR].

We fix an (ad G)-invariant nondegenerate symmetric bilinear form $(\ ,\ )_{\mathfrak{g}}$ on $\mathfrak{g}$ which restricts to a (positive definite) inner product $(\ ,\ )_{\mathfrak{k}}$ on $\mathfrak{k}$, and then using this we shall identify $\mathfrak{g}$ and $\mathfrak{k}$ with their duals respectively. Here we also assume that the semisimple part $\mathfrak{g}'$ and the center $\mathfrak{z}$ are orthogonal in $\mathfrak{g}$ with respect to $(\ ,\ )_{\mathfrak{g}}$. Now a *hyperkähler moment map* associated to the action of K on M is a K-equivariant map $\mu: M \to \mathfrak{s} \times_{\mathbb{R}} \mathfrak{k}$ such that for any $a \in \mathfrak{k}$, and any smooth vector field u on M we have

$$(d\mu(u),a)_{\mathfrak{k}} = \omega(\underline{a},u)$$

as an element of $\mathfrak{s}$, where $\underline{a}$ is the vector field on M defined by a. Then, since $d\mu$ is uniquely determined by the above condition, μ is unique up to the additions of constant maps $M \to \mathfrak{s} \otimes_{\mathbb{R}} \mathfrak{z}_{\mathfrak{k}}$, where $\mathfrak{z}_{\mathfrak{k}}$ is the center of $\mathfrak{k}$.

(2.5) For any $q = ai + bj + ck \in C$, by evaluating $\mu(x) \in \mathfrak{s} \otimes_{\mathbb{R}} \mathfrak{k} \simeq \mathfrak{s}^* \otimes_{\mathbb{R}} \mathfrak{k}$ on $q \in \mathfrak{s}$, we get a K-equivariant map $\mu_q: M \to \mathfrak{k}$ which is a moment map for the induced action of K on the Kähler manifold (M_q, g); namely, for any element $a \in \mathfrak{k}$ and any smooth vector field u on M we have $(d\mu_q(u), a) = \omega_q(\underline{a}, u)$ as a function on M. Then we may write

(2.5.1) $\mu = i \otimes \mu_i + j \otimes \mu_j + k \otimes \mu_k$.

Conversely, given moment maps μ_q, $q = i, j, k$, for the action of K on the Kähler manifolds M_q the last expression defines a hyperkähler moment map.

Suppose now that we are given holomorphic and C^∞ moment maps

$\nu_j: M \to \mathfrak{g}$ and $\mu_j: M \to \mathfrak{k}$

associated respectively to the holomorphic action of G on the holomorphic symplectic manifold $(M_j, \hat{\varphi}_j)$ (cf.(2.6.1) below) and to the C^∞ action of K on the Kähler manifold (M, ω_j), where we define

$$\hat{\varphi}_j = \varphi_{k+\sqrt{-1}i} = \omega_k + \sqrt{-1}\omega_i$$

(cf.(2.2)). Write

$$\nu_j = \mu_k + \sqrt{-1}\mu_i$$

with respect to the canonical decomposition $\mathfrak{g} = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{k}$. Then μ_k and μ_i themselves are moment maps for the corresponding Kähler actions of K, and therefore we have a hyperkähler moment map $\mu: M \to \mathfrak{s} \otimes_{\mathbb{R}} \mathfrak{k}$ by the formula (2.5.1).

(2.6) If we let K act on C trivially, then in the induced K-action on $Z = M \times C$ each element acts biholomorphically on Z by the property b) of the manifold Z. Suppose further that this action of K on Z extends to a holomorphic action of G (again inducing the identity on C). On each fiber M_q the extended action necessarily preserves the holomorphic symplectic form φ_u for any $u = e(r+\sqrt{-1}s) \in \ell_q$ and coincides with the original one for $q = j$. Consider μ as a C^∞ section of the trivial bundle $E \to M$ with fiber $s^{\mathbb{C}} \otimes_{\mathbb{C}} \mathfrak{g} = \Gamma(C, L^* \otimes_{\mathbb{C}} \mathfrak{g})$ with respect to the natural inclusion $s \otimes_{\mathbb{R}} \mathfrak{k} \subset s^{\mathbb{C}} \otimes_{\mathbb{C}} \mathfrak{g}$ (cf.(2.1)). Let $\pi: Z \to M$ be the natural projection, and ν the C^∞ section over Z of $f^* L^* \otimes_{\mathbb{C}} \mathfrak{g}$ defined to be the image of $\pi^* \mu$ with respect to the natural homomorphism

$$v: \pi^* E \to f^* L^* \otimes_{\mathbb{C}} \mathfrak{g} = \mathrm{Hom}(f^* L, \mathfrak{g});$$

explicitly, for any $u \in \ell_q$ as above

$$\nu_q(z,u) = e(\mu_r(x) + \sqrt{-1}\, \mu_s(x)) \in \mathfrak{g},$$

where $\nu_q = \nu|_{M_q}$, and $z = (x,q)$, $x \in M$, $q \in C$ as follows readily from the definitions.

(2.6.1) **Lemma**. For fixed $q \in C$ and $u \in \ell_q$, $\nu_{q,u}(x) := \nu_q(x,u)$: $M \to \mathfrak{g}$ is a holomorphic moment map for the holomorphic symplectic manifold (M_q, φ_u); namely, it is G-equivariant and satisfies the equality $(d\nu_{q,u}(v), a)_{\mathfrak{g}} = \varphi(\underline{a}, v)$ for any element $a \in \mathfrak{g}$ and any vector field v on M, where $\underline{a} = \underline{a}^q$ denotes the holomorphic vector field on M_q defined by a.

Proof. We have

$$(d\nu_{q,u}(v),a) = e\{(d\mu_r(v),a) + \sqrt{-1}(d\mu_s(v),a)\}$$
$$= e\{\omega_r(\underline{a},v) + \sqrt{-1}\omega_s(\underline{a},v)\} = \varphi_u(\underline{a},v).$$

This shows that $\nu_{q,u}$ is holomorphic on M_q and satisfies the second condition for a moment map. The the equivariancy follows by the usual argument as follows: For a fixed $x \in M$ consider a holomorphic map α: $G \to \mathfrak{g}$, $\alpha(g) = g(\nu_{q,u}(x)) - \nu_{q,u}(gx)$. Then α vanishes on K and hence on its complexfication G.