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Interpolation of Linear Operators

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ИНТЕРПОЛЯЦИЯ ЛИНЕЙНЫХ ОПЕРАТОРОВ

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ABSTRACT. The book is devoted to an important direction in functional analysis: interpolation theory for linear operators. The main methods for constructing interpolation spaces are expounded and their properties are studied. These methods allow one to look at a number of theorems and inequalities of classical analysis from a new standpoint. Interpolation theory for operators has numerous applications in Fourier series, approximation theory, partial differential equations, etc. Some of them are developed in the book.

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FOREWORD TO THE AMERICAN EDITION

This edition includes Chapter V, "Interpolation in spaces of smooth functions", written at the same time as the preceding chapters but not included in the Soviet edition for technical reasons. This chapter expounds the abstract scheme for constructing interpolation spaces by means of an unbounded operator in a Banach space, and the corresponding approximation process. Starting points of this theory were papers by J. L. Lions, Lions and J. Peetre, and P. Grisvard, which then were extended by other authors. As an application we consider only Sobolev and Besov spaces.

The reader can get acquainted with other families of spaces in the books referred to in the foreword to the Soviet edition.

The authors express their sincere gratitude to the editor of the translation, Dr. L. J. Leifman, for his penetrating remarks that helped in eliminating a number of shortcomings.

FOREWORD

The present book is devoted to the systematic exposition of a chapter in functional analysis that has appeared and developed in the past two decades and has found applications in various fields.

The basic objects of classical functional analysis were operators acting from one Banach space (or later from a topological linear space) into another. The spaces themselves were considered as given in advance. The change of this ideology was facilitated to a significant extent by the imbedding theorems of S. L. Sobolev, in which a number of fundamental theorems and inequalities of analysis were interpreted as assertions concerning the imbedding of one Banach space into another. Imbedding theorems arose in connection with problems of the theory of partial differential equations, in which for the study of smoothness of solutions a series of spaces is introduced; for the study of the behavior near the boundary of the domain or near some singular points other types of spaces are introduced, the study of values of solutions on manifolds of smaller dimension is performed in still other spaces, etc. The abundance of various spaces required a detailed study of the interrelations between these spaces. Thus a new level of abstraction appeared, on which the Banach spaces themselves are considered as elements of some category. The interpolation theory for linear operators expounded in the book is to a great extent connected with such an approach.

The first interpolation theorem in operator theory was obtained by M. Riesz in 1926 in the form of an inequality for bilinear forms. A sharpening and operator formulation of it were given by G. O. Thorin. An essential further step was the interpolation theorem of J. Marcinkiewicz (1939), whose proof was published by A. Zygmund in 1956. In the fifties important generalizations of the Riesz-Thorin and Marcinkiewicz theorems were obtained by E. M. Stein and G. Weiss. However, all these and other communications were concerned with L_p spaces or spaces similar to them. The development of general interpolation theorems for families of abstract Hilbert and Banach spaces began in 1958 independently in several countries. The first publications are due to J. L. Lions (1958-1960), E. Gagliardo (1959-1960), A. P. Calderón (1960), and S. G. Kreĭn (1960). The work of J. A. Peetre played an essential role in the sequel. Several methods have been created for obtaining interpolation theorems, which have deep interrelations. Moreover, it became clear fairly soon that the interpolation properties of spaces intermediate between two Banach spaces are consequences of the functoriality of the methods of construction. Therefore, the main emphasis has been shifted to the study of properties of intermediate interpolation spaces obtained by various methods, and to their realization. Along with this, in the work of W. Orlicz, A. P. Calderón, G. G. Lorentz, E. M. Semenov, and others deep results have been obtained concerning the interpolation of linear operators in spaces of measurable functions.

It is impossible to expound all results of interpolation theory for linear operators in one book. We have tried to illuminate only some of the main directions in its development: the real and complex methods of constructing interpolation spaces, the method of scales of Banach spaces, and interpolation in spaces of measurable functions. Supplementary information is contained in remarks and references.

In the development of interpolation theory for operators many new general notions of functional analysis have emerged. These notions and their interrelations are studied in the first chapter of the book. The exposition is based essentially on the work of N. Aronszajn and E. Gagliardo. To read this chapter one needs to know only the basic principles of functional analysis.

The second chapter, devoted to interpolation in spaces of measurable functions, makes up a significant portion of the book. It can be read independently of the first chapter, from which only the simplest definitions are needed. The chapter contains a theorem describing all interpolation spaces between L_1 and L_∞, and a theorem which is a further extension of the Marcinkiewicz theorem. The exposition is pursued as far as concrete applications, for example, the theory of orthogonal series: convergence properties of Fourier series and the basis property of a function system are studied. Moreover, the chapter contains much auxiliary material from the theory of functions which is discussed little in the literature. Decreasing rearrangements of measurable functions are studied in detail, function spaces symmetric in the sense of E. M. Semenov, and in particular, Lorentz and Marcinkiewicz spaces, are discussed (in the foreign literature similar spaces are called invariant with respect to permutations). Sharpenings of classical inequalities of analysis (the Hardy-Littlewood, Hilbert, and other inequaities) are given.

In the third chapter the theory of scales of Banach spaces, developed mainly in the publications of S. G. Kreĭn and Ju. I. Petunin, is expounded. The prerequisite material for this is contained in the first chapter. Important properties of the scales, in particular their "almost" interpolation properties are also expounded in the fourth chapter. In the last section of the third chapter properties of the classical scale of Hölder spaces important in applications are studied in detail.

In the fourth chapter two methods of constructing interpolation spaces enjoying the largest number of applications are described in detail: the method of complex interpolation proposed independently by A. P. Calderón and J. L. Lions and extensively developed by Calderón, and the method of constants and averages due to J. L. Lions and J. Peetre. The latter method is expounded in the more general form which it acquired in the work of V. I. Dmitriev (who took the most active part in writing the corresponding section). The fourth chapter can be read independently from the second and third chapters.

The book does not include interpolation theory in spaces of smooth functions and its applications.* This theory developed under the influence of the work on imbedding theorems by S. L. Sobolev, S. M. Nikol'skiĭ and their students and followers. The abstract theory did not rise immediately and easily to the level of concrete imbedding theorems obtained by special means. However, now such a theory has been created. Its exposition apparently needs another book. One can get acquainted with it partly in the book [7] by P. L. Butzer and H. Berens. It is expounded more completely in Hans Triebel's very recent book *Interpolation theory, function spaces, differential operators* (published by VEB Deutscher Verlag Wiss., Berlin, 1977, and by North-Holland, 1978).(1) One can get acquainted with the applications of this theory to the study of boundary value problems for partial differential equations in the book of J. L. Lions and E. Magenes [27] and in Triebel's book mentioned above. At the end of the book there is a bibliography covering, in addition, the indicated part of interpolation theory.

As we noted above, some parts of the book were written by V. I. Dmitriev. I. Ja. Šneĭberg provided us with invaluable help. He participated in writing §1 of Chapter IV and read a significant portion of the book. His critical remarks

**Editor's note.* For this translation a new Chapter V was added by the authors to cover this subject.

(1)The authors are grateful to Professor Triebel for making the manuscript of this book available.

enabled us to remove a number of inaccuracies and improve some proofs. The authors express their gratitude to both of them.

Finally, we thank all participants of the Voronezh seminar on interpolation theory for linear operators, and, in particular, M. Š. Braverman, A. A. Dmitriev, E. A. Pavlov, P. A. Kučment, and A. A. Sedaev, for their constant help in the preparation of the book.

The authors

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CHAPTER I

IMBEDDED, INTERMEDIATE, AND INTERPOLATION BANACH SPACES

§1. Imbedding of Banach spaces

1. *Imbedded Banach spaces.*

DEFINITION 1.1. We shall say that *a Banach space E_1 is imbedded in a Banach space E_0* if the following conditions are satisfied:

1°. $x \in E_1$ implies that $x \in E_0$.

2°. The space E_0 induces a vector space structure on E_1 coinciding with the structure of E_1.

3°. There exists a constant C_{01} such that

$$\|x\|_{E_0} \leqslant C_{01}\|x\|_{E_1} \tag{1.1}$$

for all $x \in E_1$.

The smallest possible value of the constant C_{01} in (1.1) is called the *imbedding constant of E_1 in E_0*.

Sometimes the term "imbedding" is used in a wider sense. Instead of conditions 1° and 2°, it is required that there exist an injective linear mapping j (the imbedding operator) mapping E_1 into E_0, and then condition (1.1) is written in the form $\|jx\|_{E_0} \leqslant C_{01}\|x\|_{E_1}$.

In such a situation we shall always identify E_1 with its image jE_1.

Condition 3° can be formulated in the following equivalent form: if $x_n \to x$ in E_1, then $x_n \to x$ in E_0. In this form the definition of imbedding can be carried over to topological linear spaces, and then we say that E_1 is algebraically and topologically imbedded in E_0.

DEFINITION 1.2. The space E_1 is *densely imbedded in E_0* if conditions 1°–3° hold, and also

4°. The set E_1 is dense in E_0.

The space E_1 is *compactly imbedded in* E_0 if conditions 1°–3° hold, and also

5°. Every set bounded in the norm of E_1 is relatively compact in E_0.

In what follows we shall denote the imbedding of E_1 in E_0 by the symbol $E_1 \subset E_0$, assuming that the symbol $\subset$ means not only set-theoretic inclusion, but imbedding having the properties 2° and 3°.

If the space E_1 is imbedded in E_0, then on E_1 we can introduce a new norm:

$$\|x\|^*_{E_1} = C_{01}\|x\|_{E_1}.$$

Then the space E^*_1 equipped with this norm is isomorphic with E_1, and $\|x\|_{E_0} \leqslant \|x\|^*_{E_1}$.

In connection with this we introduce the following definition.

DEFINITION 1.3. We shall say that E_1 is *normally imbedded in* E_0 if E_1 is dense in E_0 and the imbedding constant C_{01} does not exceed one, i.e. $\|x\|_{E_0} \leqslant \|x\|_{E_1}$.

We consider some examples of normalized imbedding of Banach spaces.

Let $E_0 = C(0, 1)$ be the space of continuous functions and $E_1 = C^{(1)}(0, 1)$ the space of continuously differentiable functions. Then $C^{(1)}(0, 1)$ is normally imbedded in $C(0, 1)$, so that $C^{(1)}(0, 1) \subset C(0, 1)$ and

$$\begin{aligned}\|x\|_{C(0,1)} &= \max_{t\in[0,1]} |x(t)| \leqslant \|x\|_{C^{(1)}(0,1)} \\ &= \max_{t\in[0,1]} |x(t)| + \max_{t\in[0,1]} |x'(t)|.\end{aligned}$$

Besides, by the Weierstrass theorem the set M of algebraic polynomials is dense in $C(0, 1)$, and consequently $C^{(1)}(0, 1) \supset M$ is also dense in $C(0, 1)$. Finally, by Arzelà's theorem, $C^{(1)}(0, 1)$ is compactly imbedded in $C(0, 1)$.

We may show analogously that the space $C^{(n)}(0, 1)$ of n times continuously differentiable functions is normally and compactly imbedded in $C^{(m)}(0, 1)$ if $n > m$.

Another example of normally imbedded spaces is furnished by the spaces L_p.

Let G be a bounded domain in n-space. Consider the space L_p consisting of the real-valued or complex-valued functions that are pth power summable in G $(1 \leqslant p < \infty)$. We denote by L^α the space L_p with $p = 2/(1 - \alpha)$ in which a norm is introduced by the formula

$$\|x\|_{L^\alpha} = (\text{mes } G)^{(\alpha-1)/2}\|x\|_{L_p}.$$

The space L^β is normally imbedded in L^α if $\beta > \alpha$. Indeed, $L^\alpha \supset L^\beta$ for $\alpha < \beta$, and by the Hölder inequality we have

$$\|x\|_{L^\alpha} = (\text{mes } G)^{(\alpha-1)/2}\left(\int_G |x(t)|^{2/(1-\alpha)}\, dt\right)^{(1-\alpha)/2}$$
$$\leqslant (\text{mes } G)^{(\alpha-1)/2}(\text{mes } G)^{(\beta-\alpha)/2}\left(\int_G |x(t)|^{2/(1-\beta)}\, dt\right)^{(1-\beta)/2}$$
$$= \|x\|_{L^\beta} \qquad (x \in L^\beta).$$

The set of simple measurable functions (linear combinations of characteristic functions) is dense in any space L^α, $-1 \leqslant \alpha < 1$, and therefore L^β is dense in L^α.

It is not difficult to verify that the imbedding of L^β in L^α does not have the property of compactness.

It can be shown analogously that the sequence space l_p is normally imbedded in l_q if $p < q$.

The following circumstance should be kept in mind: the space E_1 can be densely imbedded in the space E_0, but the closure of the ball of E_1 in E_0 may contain no interior points in the sense of the norm of E_0. Moreover, the following assertion is well known:

LEMMA 1.1. *If a Banach space E_1 is imbedded in a Banach space E_0 and does not coincide with it, then the closure in E_0 of any ball of E_1 is nowhere dense in E_0.*

PROOF. We assume the contrary. Let the closure $\bar{S}_1^0$ of the unit ball S_1 of E_1 contain the ball $\sigma_{2r}(x_0)$ with center at the point x_0 and radius $2r$ (in the sense of the norm of E_0). Then the point $(y - x_0)/2 \in \bar{S}_1^0$, where $y \in \sigma_{2r}(x_0)$, runs over the ball σ_r with center at zero. Consequently, the closure of any ball $\bar{S}_k^0$ contains the ball σ_{kr}. Let z be any element of σ_r. There exists an $x_1 \in S_1$ such that $\|z - x_1\|_{E_0} \leqslant r/2$. Hence there is an $x_2 \in S_{1/2}$ such that $\|z - x_1 - x_2\|_{E_0} \leqslant r/4$. Continuing this process, we construct a sequence of elements $x_n \in S_{2^{1-n}}$ such that

$$\|z - x_1 - \cdots - x_n\|_{E_0} \leqslant r2^{-n}.$$

Then $z = \sum_1^\infty x_k$, and this series converges in the norm of E_1. Therefore $z \in E_1$. We have arrived at a contradiction to the assumption that $E_1 \neq E_0$.

2. *Relative completion.* Let E_0 and E_1 be a couple of imbedded Banach spaces ($E_1 \subset E_0$). Denote by E_{01} the collection of all elements of E_0 which are limits in E_0 of sequences of elements from E_1 bounded in the norm of E_1:

$$x \in E_{01}\colon x = \lim_{n\to\infty} x_n \quad (\text{in } E_0) \quad \text{and} \quad \|x_n\|_{E_1} \leqslant R. \tag{1.2}$$

Obviously E_{01} is a linear manifold in E_0. We can introduce a norm in it by putting

$$\|x\|_{01} = \inf R,$$

where the infimum is taken over those R for which there exist sequences x_n with property (1.2). We verify that $\|x\|_{01}$ has the properties of a norm. For this we note that (1.1) implies that $\|x_n\|_{E_0} \leqslant C_{01}\|x_n\|_{E_1} \leqslant C_{01}R$, and since $x_n \to x$ in E_0, we also have $\|x\|_{E_0} \leqslant C_{01}R$. This implies that $\|x\|_{E_0} \leqslant C_{01}\|x\|_{01}$, and, in particular, that $\|x\|_{01} = 0$ only if $x = 0$. It is obvious that $\|\lambda x\|_{01} = |\lambda|\ \|x\|_{01}$. Moreover, if the sequences $\{x_n\}$ and $\{y_n\}$ have properties (1.2) relative to the elements x and y, respectively, with constants R and R_1, then $x_n + y_n \to x + y$ in E_0, and $\|x_n + y_n\|_{E_1} \leqslant R + R_1$.

We obtain from this that $\|x + y\|_{01} \leqslant R + R_1$, and then, taking the infimum on the right side, that $\|x + y\|_{01} \leqslant \|x\|_{01} + \|y\|_{01}$.

We note that for $x \in E_1$ we may choose $x_n \equiv x$, and from the definition of the norm in E_{01} it will follow that $\|x\|_{01} \leqslant \|x\|_{E_1}$.

The construction of E_{01} may be ascribed the following geometrical meaning. Let $x \in E_{01}$ and $\|x\|_{01} = r$. By definition, the element x belongs to the closure in E_0 of any ball of E_1 with radius $R > r$. Taking a sequence $R_k \to r$ and for every R_k, choosing an appropriate sequence of elements in E_1 with property (1.2), we can construct a sequence $\{x_k'\} \subset E_1$ such that $x_k' \to x$ in E_0 and $\|x_k'\|_{E_1} \to r$ as $k \to \infty$. Then $\bar{x}_k = rx_k'(\|x_k'\|_{E_1})^{-1} \to x$ in E_0, and $\|\bar{x}_k\|_{E_1} = r$. Thus, x belongs to the closure in E_0 of the ball (and even of the sphere) of radius r of E_1 and does not belong to the closure of balls of smaller radius.

Hence, the ball of E_{01} is the closure in E_0 of the ball of E_1 with the same radius.

We prove that the normed space E_{01} is complete.

Let $\{x^{(k)}\}$ be a Cauchy sequence in E_{01}. By the inequality $\|x\|_{E_0} \leqslant C_{01}\|x\|_{01}$, it is Cauchy in E_0. Let $x^{(k)} \to x$ in E_0. For any $\varepsilon > 0$ and sufficiently large m and l we have $\|x^{(m)} - x^{(l)}\|_{01} \leqslant \varepsilon$. This means that $x^{(m)} - x^{(l)}$ belongs to the closure in E_0 of the ball of radius ε of the space E_1. However, $x^{(m)} - x^{(l)} \to x^{(m)} - x$ in E_0 as $l \to \infty$. Therefore, the element $x^{(m)} - x$ belongs to the same closure, i.e. $\|x^{(m)} - x\|_{01} \leqslant \varepsilon$. Thus, $x^{(k)} \to x$ in E_{01} as $k \to \infty$, and E_{01} is complete.

Summing up, we may say that we have constructed a Banach space E_{01} such that $E_1 \subset E_{01} \subset E_0$; moreover, $\|x\|_{01} \leqslant \|x\|_{E_1}$ $(x \in E_1)$ and

$$\|x\|_{E_0} \leqslant C_{01}\|x\|_{01} \qquad (x \in E_{01}). \tag{1.3}$$

From (1.3) it follows that if E_1 is normally imbedded in E_0, so is E_{01}.

The Banach space E_{01} is called the *relative completion of* E_1 *with respect to* E_0.

The following facts are important.

LEMMA 1.2. *If E_1 does not coincide with E_0, then its relative completion E_{01} does not coincide with E_0 either.*

PROOF. By Lemma 1.1 the closure in E_0 of any ball S_r of E_1 is nowhere dense in E_0. Therefore $E_{01} = \bigcup_{r>0} S_r$ is a set of the first category in E_0, and consequently does not coincide with E_0.

LEMMA 1.3. *The completion $\hat{E}_{01}$ of E_{01} with respect to E_0 coincides with E_{01} itself.*

PROOF. If $y \in \hat{E}_{01}$, then there is a sequence of elements $y_k \in E_{01}$ such that $\|y_k\|_{01} = \|y\|_{\hat{E}_{01}}$ and $y_k \to y$ in E_0. For every y_k there exists a sequence $x_n^{(k)} \to y_k$ in E_0 as $n \to \infty$ and such that $\|x_n^{(k)}\|_{E_1} = \|y_k\|_{E_{01}} = \|y\|_{\hat{E}_{01}}$. But then we can construct a sequence $x_{n_k}^{(k)}$ such that $x_{n_k}^{(k)} \to y$ in E_0 and $\|x_{n_k}^{(k)}\|_{E_1} = \|y\|_{\hat{E}_{01}}$, and consequently $y \in E_{01}$. Moreover, from the above it follows that $\|y\|_{01} \leqslant \|y\|_{\hat{E}_{01}}$, and since the reverse inequality is always true (see (1.3)), we have $\|y\|_{01} = \|y\|_{\hat{E}_{01}}$. The lemma is proved.

A simple example of relative completion can be obtained by setting $E_0 = L_1(0, 1)$ and $E_1 = C(0, 1)$. Then it is easy to see that $E_{01} = L_\infty(0, 1)$. Similarly, if $E_0 = C(0, 1)$ and $E_1 = C_1(0, 1)$, then $E_{01} = H_1(0, 1)$ is the space of functions satisfying a Lipschitz condition.

The following assertions concerning the relation between the spaces E_1 and E_0 are consequences of the definition of relative completion.

LEMMA 1.4. *In order that E_1 be isometrically imbedded in E_{01} it is necessary and sufficient that the ball of E_1 be closed (in E_1) in the topology induced by the norm of E_0.*

PROOF. If $\|x\|_{01} \leqslant \|x\|_{E_1}$ for some $x \in E_1$, then there exists a sequence $x_n \to x$ in E_0 such that $\|x_n\|_{E_1} = \|x\|_{01} = a$. This means that the ball of radius a in E_1 is not closed in the norm of E_0. Its limit point x has norm greater than a.

Conversely, if the ball of radius a is not closed, then there exists a point x with $\|x\|_{E_1} > a$ and a sequence $x_n \to x$ in E_0 such that $\|x_n\|_{E_1} \leqslant a$. But then $\|x\|_{01} \leqslant a < \|x\|_{E_1}$.

LEMMA 1.5. *In order that E_1 be a closed subset of E_{01} it is necessary and sufficient that the closure of a ball of E_1 in the topology induced in E_1 by the norm of E_0 be a bounded set in E_1.*

By the Banach theorem, E_1 is closed in E_{01} if and only if the norms $\|x\|_{E_1}$ and $\|x\|_{01}$ are equivalent on E_1. After this remark the proof of Lemma 1.5 can be carried out similarly to that of Lemma 1.4.

DEFINITION 1.4. The space E_1 is said to be *complete with respect to* E_0 if E_{01} coincides with E_1 isometrically.

By Lemma 1.3, E_{01} is complete with respect to E_0. In particular, $L_\infty(0, 1)$ is complete with respect to $L_1(0, 1)$.

LEMMA 1.6. *If $E_1 \subset F \subset E_0$, then the completion $E_{F,1}$ of E_1 with respect to F is imbedded in E_{01} with imbedding constant not exceeding one. The completion of $E_{F,1}$ with respect to E_0 coincides with E_{01} isometrically.*

PROOF. If $x \in E_{F,1}$, then there exists a sequence $x_n \in E_1$ with $\|x_n\|_{E_1} = \|x\|_{E_{F,1}}$ such that $x_n \to x$ in F. In view of the imbedding $F \subset E_0$ we then have $x_n \to x$ in E_0, and consequently $x \in E_{01}$ and $\|x\|_{E_{01}} \leqslant \|x\|_{E_{F,1}}$.

Moreover, the unit ball of the completion of $E_{F,1}$ with respect to E_0 is the closure in E_0 of the unit ball of $E_{F,1}$; the unit ball of E_1 is dense in the latter ball in the norm of F, and consequently in the norm of E_0 as well. Thus, the unit ball of the completion of $E_{F,1}$ with respect to E_0 coincides with the closure in E_0 of the unit ball of E_1, i.e. with the unit ball of E_{01}.

COROLLARY 1. *If E_1 is complete with respect to E_0, then it is complete with respect to F.*

For example, $L_\infty(0, 1)$ is complete with respect to all spaces $L_p(0, 1)$, $1 \leqslant p < \infty$.

COROLLARY 2. *If $E_{F,1}$ is complete with respect to E_0, then the completion of $E_{F,1}$ coincides with that of E_{01}.*

§2. Dual spaces of imbedded Banach spaces

1. *Dual spaces and relative completion.* If the space E_1 is imbedded in the space E_0, then the restriction to E_1 of every continuous linear functional $f(x)$ defined on E_0 induces a functional on E_1 in a natural manner. This functional is continuous in the norm of E_1. Indeed,

$$|f(x)| \leqslant \|f\|_{E_0'} \|x\|_{E_0} \leqslant C_{01}\|f\|_{E_0'} \|x\|_{E_1}.$$

Thus, a linear mapping of the space E_0' into E_1' is obtained. If E_1 is not dense in E_0, then there exists a nonzero functional in E_0' which identically vanishes on E_1, and consequently is mapped into zero. In this case the mapping is not injective. If, on the other hand, E_1 is dense in E_0, then the mapping is injective, and E_0' can be considered imbedded in E_1'. Then, the imbedding constant does not exceed C_{01}.

The quantity

$$\sup_{x \in E_1} \frac{|f(x)|}{\|x\|_{E_1}} = \|f\|_{E_1'} \qquad (f \in E_0')$$

will be a seminorm on E_0' in the first case and a norm on E_1' in the second case.

Now we give another characterization of the relative completion E_{01}.

LEMMA 2.1. *The completion E_{01} of E_1 with respect to E_0 consists of all elements of E_0 inducing functionals on E_0', bounded in the seminorm $\|f\|_{E_1'}$, according to the formula $x(f) = f(x)$.*

PROOF. If $x \in E_{01}$, then there exists a sequence x_n such that $x_n \to x$ in E_0 and $\|x_n\|_{E_1} = \|x\|_{01}$. Let $f \in E_0'$. We have $f(x_n) \to f(x)$ and $|f(x_n)| \leqslant \|f\|_{E_1'} \|x_n\|_{E_1} = \|f\|_{E_1'} \|x\|_{01}$. This implies that $|f(x)| \leqslant \|f\|_{E_1'} \|x\|_{01}$ also holds. Thus, the functional $x(f)$ is bounded on E_0' in the seminorm $\|f\|_{E_1'}$, and its norm does not exceed $\|x\|_{01}$.

Now let $x \in E_0$ and $|x(f)| = |f(x)| \leqslant C\|f\|_{E_1'}$ $(f \in E_0')$. We show that x belongs to the closure in E_0 of the ball S_C of radius C of E_1. Otherwise there would exist a functional $f_0 \in E_0'$ such that $\sup_{y \in S_C}|f_0(y)| < f_0(x)$ or $C\|f_0\|_{E_1'} < f_0(x)$, which contradicts the initial assumption. Thus, $x \in E_{01}$ and $\|x\|_{01} \leqslant C$.

From the proof of the lemma we obtain the following corollary.

COROLLARY 1. *The norm of the functional $x(f)$ with respect to the seminorm $\|f\|_{E_1'}$ is equal to $\|x\|_{01}$:*

$$\|x\|_{01} = \sup_{f \in E_0', \|f\|_{E_1'} \leqslant 1} |f(x)|. \tag{2.1}$$

DEFINITION 2.1. A linear manifold M' of continuous linear functionals on a Banach space E is said to be *normative* if

$$\|x\|_E = \sup_{f \in M', \|f\|_{E'} \leqslant 1} |f(x)| \qquad (x \in E).$$

Formula (2.1) and Lemma 1.4 imply the following theorem.

THEOREM 2.1. *In order that the restrictions to E_1 of all functionals from E_0' form a normative set for E_1 it is necessary and sufficient that E_1 be imbedded isometrically in E_{01}, or, what is the same, a ball of E_1 be closed in E_1 in the topology induced by the norm of E_0.*

2. *Dual spaces of densely imbedded spaces.* If E_1 is densely imbedded in E_0, then, as we have seen above, E_0' is imbedded in E_1'; however, this imbedding may not be dense. For example, $l_1 = E_1$ is normally imbedded in $c_0 = E_0$.

The dual spaces $E_0' = l_1$ and $E_1' = l_\infty$ are not densely imbedded. Indeed, the element $(1, 1, \dots) \in l_\infty$ is at distance one from the linear manifold l_1.

We recall that a linear manifold $M' \subset E_1'$ is said to be *total* if the condition $f(x_0) = 0$ $(x_0 \in E_1)$ for all $f \in M'$ implies that $x_0 = \theta$.[1] If E_1 is dense in E_0, then E_0' is a total linear manifold in E_1'. Indeed, if $x_0 \in E_1$ and $x_0 \neq 0$, then $x_0 \in E_0$, and so there is a functional $f_0 \in E_0'$ such that $f_0(x_0) = \|x_0\|_{E_0} \neq 0$.

If E_1 is reflexive, then every total manifold is dense in E_1', and so in this case E_0' is densely imbedded in E_1'.

THEOREM 2.2. *If E_1 is densely imbedded in E_0, then E_0' is complete with respect to E_1'.*

PROOF. If f belongs to the completion $(E_0')_1$ of E_0' with respect to E_1', then there exists a sequence $f_n \in E_0'$ such that $f_n \to f$ in E_1' and $\|f_n\|_{E_0'} = \|f\|_{(E_0')_1}$. Then for any $x \in E_1$ we have $f_n(x) \to f(x)$. The sequence of the linear functionals $f_n \in E_0'$ is uniformly bounded and converges to f on the set E_1 dense in E_0. By the Banach-Steinhaus theorem this implies that $f \in E_0'$ and the sequence f_n weakly converges to f on E_0. Moreover, $\|f\|_{E_0'} \leqslant \underline{\lim}\|f_n\|_{E_0'} = \|f\|_{(E_0')_1}$. Since in the case of a completion the reverse inequality always holds, we have $\|f\|_{E_0'} = \|f\|_{(E_0')_1}$.

THEOREM 2.3. *A reflexive space E_1 is complete with respect to any other space E_0 in which it is imbedded.*

PROOF. Without loss of generality we may assume that E_1 is dense in E_0, since otherwise E_0 could be replaced by the closure of E_1 in E_0 without changing the completion E_{01}. Under this assumption, the reflexivity of E_1 implies that E_0' is densely imbedded in E_1'. Then by Theorem 2.2 the space $E_1'' = E_1$ is complete with respect to E_0'', and by Corollary 1 of Lemma 1.6 the space E_1 is complete with respect to E_0. The theorem is proved.

If E_0' is densely imbedded in E_1', then it is a normative set on E_1, and therefore Lemma 2.1 and formula (2.1) imply the following assertion.

LEMMA 2.2. *If E_1 is densely imbedded in E_0 and E_0' is densely imbedded in E_1', then E_1 is isometrically imbedded in E_{01}. The space E_{01} can be isometrically imbedded in E_1'' in a natural manner, and in this natural identification we may assume that $E_{01} = E_0 \cap E_1''$.*

Now if E_0' is not densely imbedded in E_1', we can consider its closure $\mathring{E}_1'$ in E_1'. This is a subspace of E_1'. In this case we may reformulate Lemma 2.1 and Corollary 1 as follows.

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[1] θ is the origin of the space.