Lecture Notes in Mathematics

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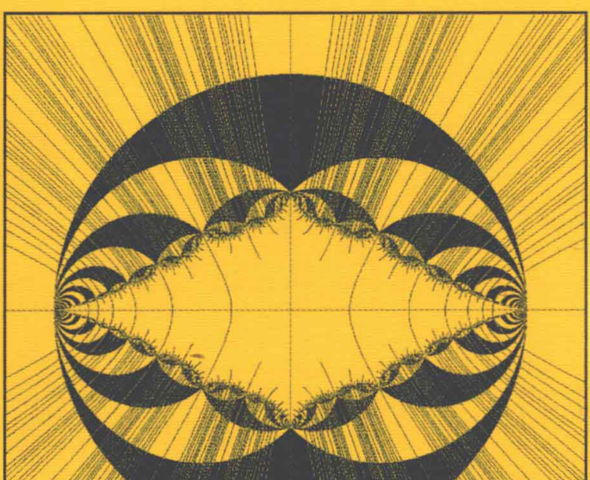
Punctured Torus Groups and 2-Bridge Knot Groups I



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Punctured Torus Groups and 2-Bridge Knot Groups (I)

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Lecture Notes in Mathematics 1909

Preface

The main purpose of this monograph is to give a full description of Jorgensen's theory on the space $\mathcal{QF}$ of quasifuchsian (once) punctured torus groups with a complete proof. Our method is based on Poincare's theorem on fundamental polyhedra. This geometric approach enabled us to extend Jorgensen's theory beyond the quasifuchsian space and apply to knot theory.

1. History

By the late 70's Troels Jorgensen had made a series of detailed studies on the space $\mathcal{QF}$ of quasifuchsian (once) punctured torus groups from the view point of their Ford fundamental domains. These studies are summarized in his famous unfinished paper [40]. In it, he gave a complete description of the combinatorial structure of the Ford domain of every quasifuchsian punctured torus group, and showed that the space $\mathcal{QF}$ can be described in terms of the combinatorics of the faces of the Ford domain. This led to the description of $\mathcal{QF}$ in terms of the Farey triangulation, or the modular diagram. As a byproduct, the first examples of surface bundles over the circle with complete hyperbolic structures were obtained (cf. [41] and [43]).

To date, most of Jorgensen's work has not been published, yet it became widely known, motivated various research projects, and was successfully applied. His work, together with Riley's construction [67] of the complete hyperbolic structure on the figure-eight knot complement, has motivated Thurston's uniformization theorem of surface bundles over the circle [77] (cf. [63]). It had also motivated the experimental study by Mumford, McMullen and Wright [60] of the limit sets of quasifuchsian punctured torus groups. This work was sublimated into the beautiful book [61] by Mumford, Series and Wright, which displays deeply hidden fractal shapes of the space $\mathcal{QF}$ and the limit sets of punctured torus Kleinian groups.

2. Motivation

The authors' interest in Jorgensen's work grew from knot theory. We are interested in hyperbolic knots, and more generally hyperbolic links, i.e., mutually disjoint circles embedded in the 3-sphere S^3 whose complements admit complete hyperbolic structures of finite volume. Recall that the Ford domain of a complete cusped hyperbolic manifold of finite volume is the geometric dual to the canonical ideal polyhedral decomposition introduced by Epstein and Penner [27] (cf. [81]). Thus, by virtue of Mostow rigidity, the combinatorial structure of the Ford domain is a complete invariant of the topological type of such a manifold. In particular, by the knot complementary theorem due to Gordon and Luecke [32], this gives a complete invariant of a hyperbolic knot. In the joint work [71] with Weeks, the second author gave certain topological decompositions of 2-bridge link complements into topological ideal tetrahedra, by imitating Jorgensen's decomposition of punctured torus bundles over the circle (cf. [29]), and conjectured that they are combinatorially equivalent to the canonical decompositions. Here, a 2-bridge link is a link which can be drawn with only two local maxima and minima in the vertical direction (see Fig. 0.1). We had thought that if we could understand Jorgensen's work, then we would be able to prove the conjecture.

3. Extending of Jorgensen's theory beyond the quasifuchsian space and application to 2-bridge links

Fortunately, this turned out to be the case. Namely, we found a very natural way to understand the hyperbolic structures and the canonical decompositions of the 2-bridge link complements in the context of Jorgensen's work. To describe the idea, recall that the 2-bridge links are parametrized by pairs (p, q) of relatively prime integers (see [22, Chap. 12]) and that the complement of the 2-bridge link of type (p, q) is homeomorphic to (the interior of) the manifold obtained from $S \times [-1, 1]$, with S a 4-times punctured sphere, by attaching 2-handles along $\alpha \times (-1)$ and $\beta \times 1$, where α and β are simple loops on S of slopes $1/0$ and q/p, respectively (see Sect. 2.1, p. 16, for the definition of a slope); in particular, the link group (i.e., the fundamental group of the complement of the link) is isomorphic to the quotient group $\pi_1(S)/\langle\langle \alpha, \beta \rangle\rangle$, where $\langle\langle \cdot \rangle\rangle$ denotes the normal closure. The extended Jorgensen's theory realizes the operation of attaching 2-handles by a continuous family of hyperbolic cone-manifolds, whose cone axes are the union of the *upper* and *lower tunnels*, i.e., the co-cores of the 2-handles (see Fig. 0.1).

According to Keen-Series' theory of pleating varieties [44, 45, 46, 47, 49], $\mathcal{QF}$ is foliated by the pleating varieties, $\mathcal{P}(\lambda^-, \lambda^+)$, where (λ^-, λ^+) runs over (ordered) pairs of distinct projective measured laminations of the punctured torus T. By extending Jorgensen's theory beyond the quasifuchsian space (cf.

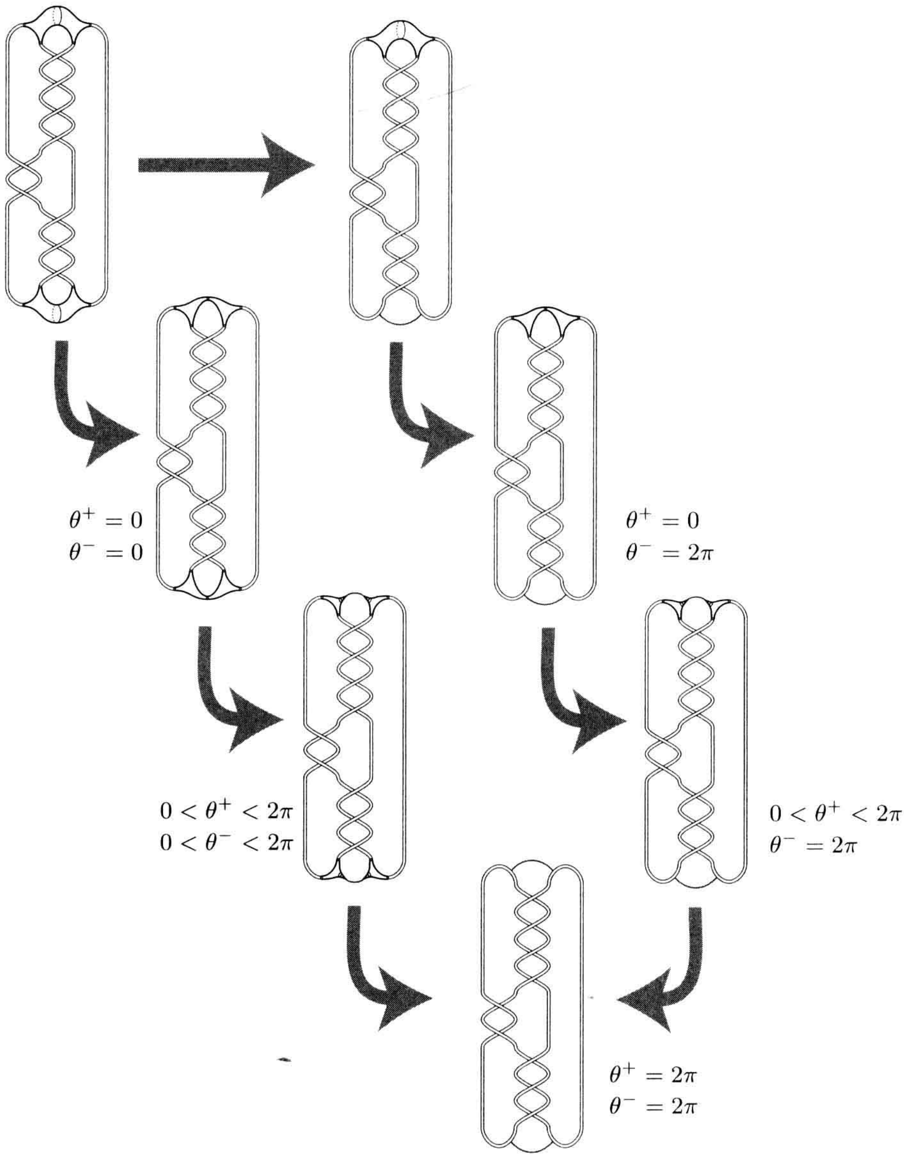

Fig. 0.1. Continuous family of hyperbolic cone-manifolds $M(\theta^-, \theta^+)$

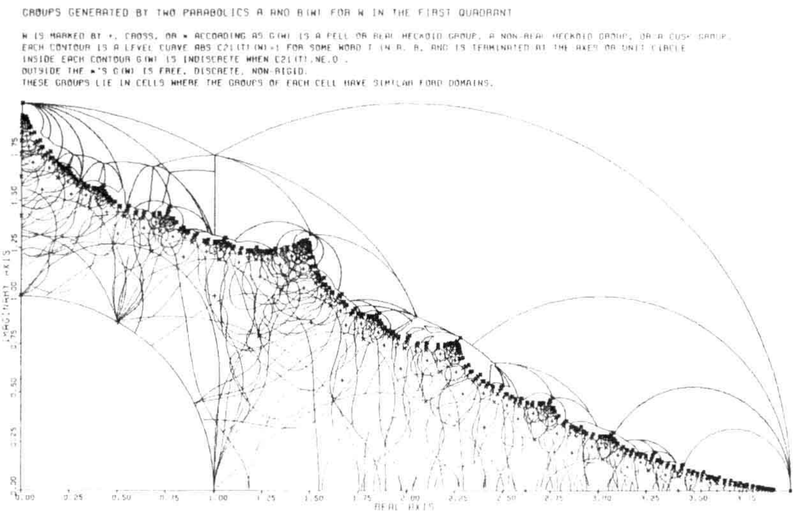

Fig. 0.2a. Riley's pioneering exploration of groups generated by two parabolic transformations. This computer-drawn picture has been circulated among the experts and has inspired many researchers in the fields of Kleinian groups and knot theory. This specific copy of the picture was obtained directly from Prof. Riley by the third author when he visited SUNY Binghamton in February 1991.

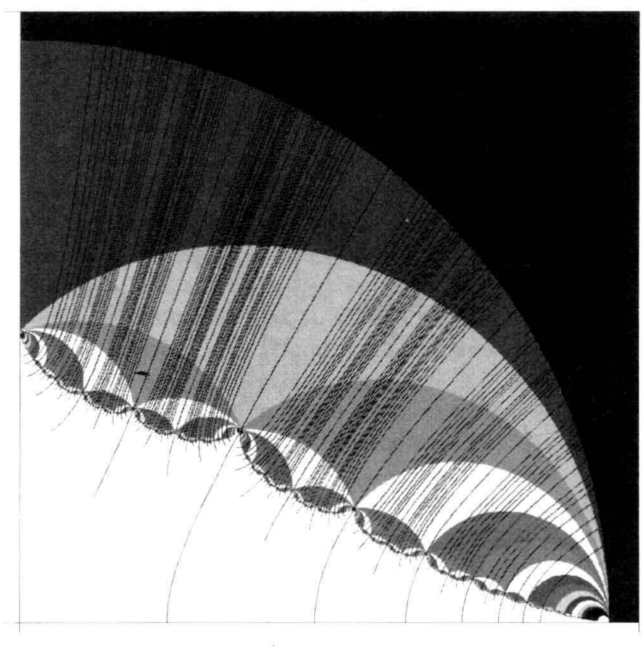

Fig. 0.2b. Riley slice of Schottky space together with pleating rays and their extensions

[66]), we found that if (λ^-, λ^+) is rational, i.e., each of $\lambda^\pm$ are rational, then the following hold:

1. The pleating variety $\mathcal{P}(\lambda^-, \lambda^+)$ has a natural extension to the outside of $\mathcal{QF}$ in the space of type-preserving representations of the fundamental group $\pi_1(T)$.
2. Each point in the extension is the holonomy representation of a certain hyperbolic cone manifold, which is commensurable with the hyperbolic cone manifold, $M(\theta^-, \theta^+)$, whose underlying space is the complement of a 2-bridge link and whose singular set is the union of the upper and lower tunnels, which have the cone angles θ^+ and θ^-, respectively. Moreover the 2-bridge link is of type (p, q), or of slope q/p, if (λ^-, λ^+) is equivalent to $(1/0, q/p)$ by a modular transformation.
3. If the (edge path) distance $d(1/0, q/p)$ in the Farey triangulation is ≥ 3, namely if $q \not\equiv \pm 1 \pmod p$, then the hyperbolic cone manifold $M(\theta^-, \theta^+)$ exists for every pair of cone angles in $[0, 2\pi]$. Thus we have a continuous family of hyperbolic cone manifolds connecting $M(0,0)$, the quotient hyperbolic manifold of a doubly cusped group, with $M(2\pi, 2\pi)$, the complete hyperbolic structure of the 2-bridge link complement.
4. If $1 \leq d(1/0, q/p) \leq 2$, namely if $q \equiv \pm 1 \pmod p$ and $p \neq 0$, then the hyperbolic cone manifold $M(\theta^-, \theta^+)$ exists for every pair of cone angles in $[0, 2\pi]$, except the pair $(2\pi, 2\pi)$. In addition, if $p \geq 3$, $M(\theta^-, \theta^+)$ collapses to the base orbifold of the Seifert fibered structure of the link complement as both cone angles approach 2π.
5. The holonomy group of $M(\theta^-, \theta^+)$ is discrete if and only if $\theta^\pm \in \{2\pi/n \,|\, n \in \mathbb{N}\} \cup \{0\}$. In particular, that of $M(2\pi, 2\pi/n)$ is generated by two parabolic transformations, which Riley called a *Heckoid group* in [68].

Actually, we have constructed these hyperbolic cone manifolds by explicitly constructing "Ford fundamental polyhedra". In other words, we have extended Jorgensen's description of the Ford fundamental polyhedra for quasifuchsian punctured torus groups to those of the hyperbolic cone manifolds arising from the 2-bridge links. In particular, we have shown that the canonical decompositions of hyperbolic 2-bridge link complements are isotopic to the topological ideal tetrahedral decompositions constructed in [71], proving the conjecture which motivated our project.

The above result also enables us to locate the 2-bridge link groups in the representation space (Fig. 0.2b). The shaded region of the figure illustrates (the first quadrant of) the *Riley slice* of the Schottky space, i.e., the subspace of $\mathbb{C}$ consisting of those complex numbers ω such that the group

$$G_\omega = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \right\rangle$$

is discrete and free and such that the quotient $\Omega(G_\omega)/G_\omega$ of the domain of discontinuity is homeomorphic to the 4-times punctured sphere S (Definition

5.3.5). Each shaded region represents groups whose Ford domains have the same combinatorics. The lines in the shaded region are pleating rays of the Riley slice ([45]) and their extensions to the outside of the Riley slice correspond to the hyperbolic cone-manifolds $M(2\pi, \theta)$. In particular the end-points with positive imaginary parts represent hyperbolic 2-bridge link groups and those on the real line represent the orbifold fundamental groups of the base orbifolds of the Seifert fibered structures of non-hyperbolic 2-bridge link complements.

We think this realizes what Riley had in mind, for he devoted time and effort to identify 2-bridge link groups in the space of non-elementary two parabolic groups, yielding the mysterious output in Fig. 0.2a ([69]).

This describes a relation between the hyperbolic structure and the bridge structure of a 2-bridge link complement. Since a bridge structure is a kind of Heegaard structure, it is naturally expected that a similar relation holds between the hyperbolic structures and the Heegaard structures of hyperbolic manifolds. In particular, we conjecture that this is the case for tunnel number 1 hyperbolic knots and their unknotting tunnels. An *unknotting tunnel* for a knot K is an arc τ in S^3 with $\tau \cap K = \partial\tau$ such that the complement of an open regular neighborhood is homeomorphic to a genus 2 handlebody. A knot which admits an unknotting tunnel is said to have *tunnel number* 1. For example, a 2-bridge knot has tunnel number 1 and each of the upper and lower tunnels is an unknotting tunnel. Tunnel number 1 knots have been extensively studied, and in particular, non-hyperbolic tunnel number 1 knots were classified by [59]. An unknotting tunnel τ of a tunnel number 1 knot K gives a *Heegaard structure* of the knot complement $S^3 - K$, in the sense that $S^3 - K$ is homeomorphic to (the interior) of the manifold obtained from the genus 2 handlebody by adding a 2-handle, where τ corresponds to the co-core of the 2-handle. We would like to propose the following conjecture.

Conjecture. Let K be a tunnel number 1 hyperbolic knot and let τ be an unknotting tunnel for K. Then there is a continuous family of hyperbolic cone manifolds whose underlying space is the knot complement and whose cone axis is the unknotting tunnel τ, where the cone angle varies from 0 to 2π. In particular, τ is isotopic to a geodesic in the hyperbolic manifold $S^3 - K$.

4. Related results

Some of these results were announced in [8, 9, 10], and our original plan was to write a single paper or a book which contains the whole story. However, we found it very difficult to explain the whole theory at once, and thus decided to divide it into a few papers. This monograph is the first part of the series, and its main purpose is to give a full description of Jorgensen's theory on the space $\mathcal{QF}$ with a complete proof.

For Jorgensen's theory on the space $\mathcal{QF}$, supervised by Dunfield and partially influenced by [9] and [78], Schedler [72] gave a treatment based on the theory of holomorphic motions. Though the bijectivity of the *side parameter* map is not proved in his paper, his approach using holomorphic motions is natural and further development is expected in the future.

Our approach in turn is based on Poincare's theorem on fundamental polyhedra. This geometric approach enables us to extend Jorgensen's theory beyond the quasifuchsian space, where we need to treat indiscrete groups.

For (attempts of) expositions of Jorgensen's theory without proof, see [75, 8, 9, 65, 70].

The first author has extended Jorgensen's theory to the closure of $\mathcal{QF}$ in [2]. In particular, a rigorous proof was given to the well-known description of the Ford domain of the punctured torus bundles over the circle (cf. [12, 64]). We note that Lackenby [52] gave a topological proof to the fact that Jorgensen's ideal triangulations of punctured torus bundles are genuine geometric decompositions. Gueritaud [33] also gave an alternative proof to this fact by using the angle structure. In the appendix of the paper, Futer proves by modifying Gueritaud's argument that the topological ideal triangulations of the 2-bridge link complements in [71] are also geometric. Moreover, Gueritaud [34] also proved that these geometric decompositions are canonical.

In [3], the first author has found a nice relation between Jorgensen's parameter of $\mathcal{QF}$ and the conformal end invariant of elements of $\mathcal{QF}$. This together with Brock's results [21] leads to an estimate of the convex core volume in terms of Jorgensen's parameter. He has also found interesting applications of Jorgensen's theory to knot theory in [4].

The computer program, OPTi [78] (cf. [79]), has been developed by the third author for the project, and it has been a driving force for our work. It is our pleasure that it has now become a favorite tool for various colleagues in the world.

Collaborating with Komori and Sugawa, the third and last authors launched a project to draw Bers' slices of $\mathcal{QF}$, and various mysterious pictures have been produced ([50] and [82]).

5. A quick trip through Jorgensen's theory and its generalization

Jorgensen's theory enables us to intuitively understand how a simple fuchsian group evolves into complicated quasifuchsian punctured torus groups and boundary groups, by looking at their Ford domains (see Figs. 0.3–0.10, 0.17, 0.19–0.21 and 1.2). Jorgensen expresses this phenomenon as follows. *The Ford domain records the history of how the quasifuchsian group evolved from a simple fuchsian group.*

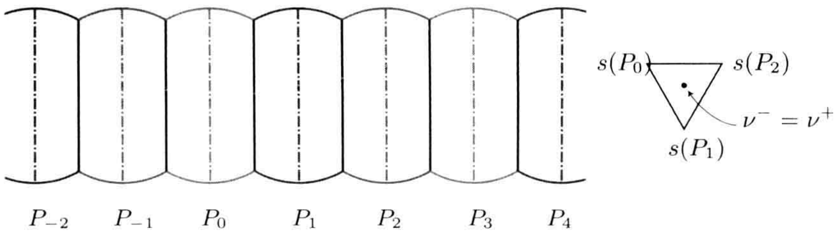

Fig. 0.3. $(1/3, 1/3, 1/3)$

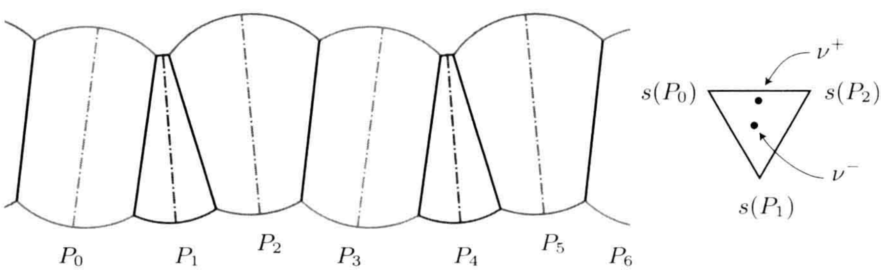

Fig. 0.4. $(0.421397 - 0.0483593\mathrm{i}, 0.295605 - 0.0422088\mathrm{i}, 0.282998 + 0.0905681\mathrm{i})$

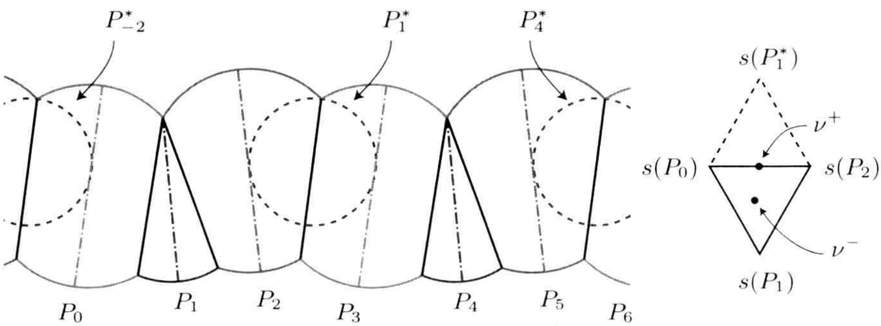

Fig. 0.5. $(0.433791 - 0.0551654\mathrm{i}, 0.290295 - 0.0481496\mathrm{i}, 0.275914 + 0.103315\mathrm{i})$

5.1. A fuchsian punctured torus group

The starting point of Jorgensen's theory is the fuchsian group illustrated in Fig. 0.11. For each integer j, let L_j be the geodesic in the upper half plane model $\mathbb{H}^2$ of the hyperbolic plane, represented by the Euclidean half circle with center $j/3$ and radius $1/3$. Let P_j be the order 2 elliptic transformation whose fixed point is equal to the highest point $(j + \mathrm{i})/3$ of L_j where $\mathrm{i} = \sqrt{-1}$. Then P_j interchanges the inside and outside of L_j and acts on L_j as a Euclidean isometry. The product $P_{j+2}P_{j+1}P_j$ is equal to the parabolic transformation $K(z) = z + 1$. Note that $P_{j+3n} = K^n P_j K^{-n}$ for every j, $n \in \mathbb{Z}$. Let Γ be

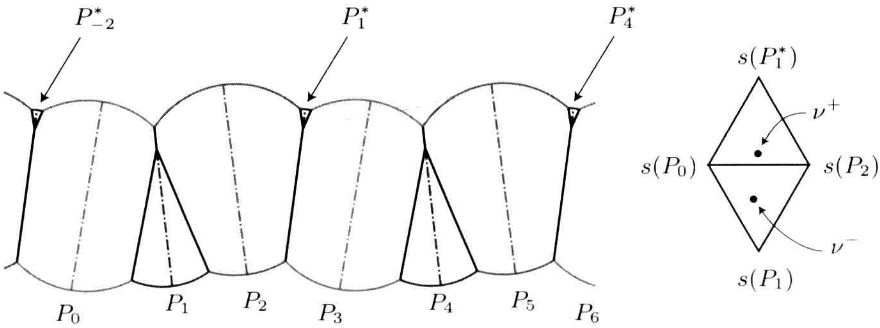

Fig. 0.6. $(0.444228 - 0.0608968\mathrm{i}, 0.285823 - 0.0531522\mathrm{i}, 0.269949 + 0.114049\mathrm{i})$

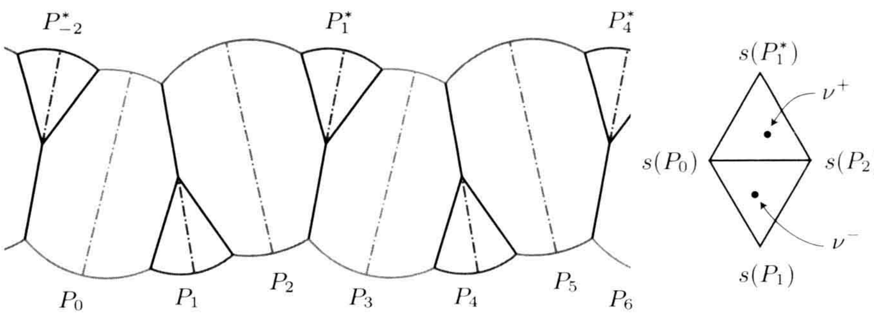

Fig. 0.7. $(0.496414 - 0.0895542\mathrm{i}, 0.263465 - 0.0781648\mathrm{i}, 0.240121 + 0.167719\mathrm{i})$

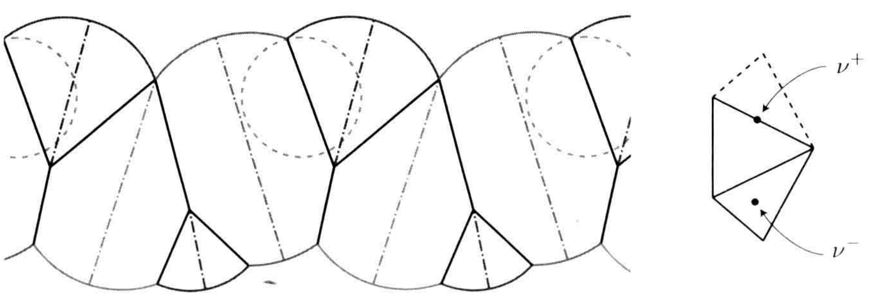

Fig. 0.8. $(0.549741 - 0.118838\mathrm{i}, 0.240619 - 0.103725\mathrm{i}, 0.20964 + 0.222563\mathrm{i})$

the group generated by $\{P_j \mid j \in \mathbb{Z}\}$. Then it is generated by three successive elements, say P_0, P_1 and P_2. Consider the shaded region R in Fig. 0.11. Then the edges of R are paired by P_0, P_1, P_2 and K. By applying Poincare's theorem on fundamental polyhedra to this setting, we see that R is a fundamental domain of the group Γ and

$$\Gamma \cong \langle P_0, P_1, P_2 \mid P_0^2 = P_1^2 = P_2^2 = 1 \rangle \cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}).$$

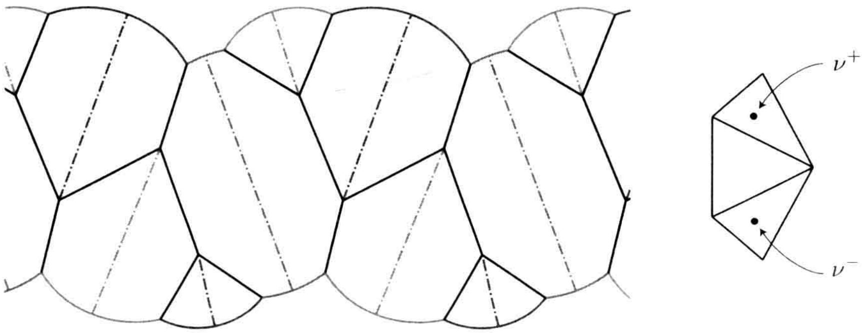

Fig. 0.9. (0.594262 − 0.143287i, 0.221545 − 0.125063i, 0.184193 + 0.26835i)

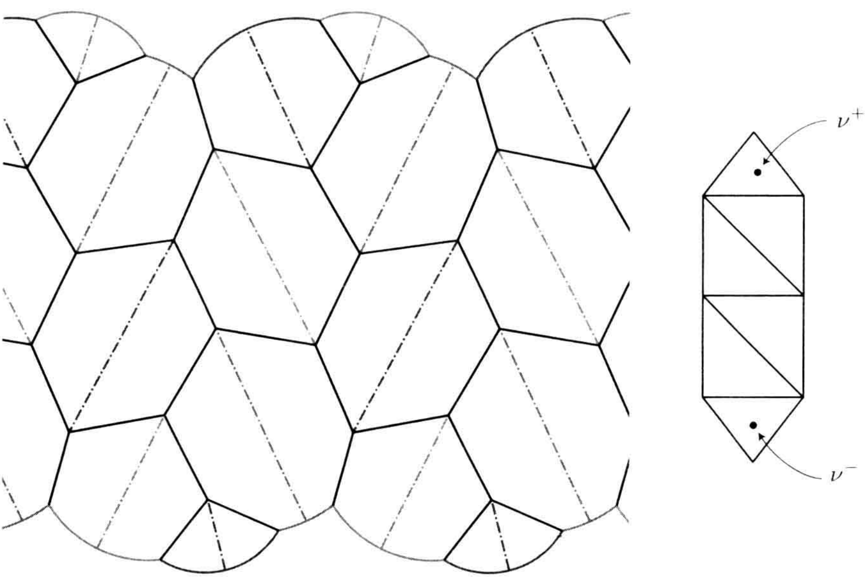

Fig. 0.10. (0.652971 − 0.175526i, 0.196392 − 0.153203i, 0.150637 + 0.328729i)

As shown in Fig. 0.12, the quotient $\mathbb{H}^2/\Gamma$ is a hyperbolic orbifold, $\mathcal{O}$, with underlying space once-punctured sphere and with three cone points of cone angle π. The subgroup Γ_0 of Γ of index 2, obtained as the kernel of the homomorphism $\Gamma \to \mathbb{Z}/2\mathbb{Z}$ sending each generator P_j to the generator of $\mathbb{Z}/2\mathbb{Z}$, is a rank 2 free group generated by $A := KP_0 = P_2P_1$ and $B := K^{-1}P_2 = P_0P_1$. The union $R \cup K(R)$ is a fundamental domain of Γ_0, and the quotient $\mathbb{H}^2/\Gamma_0$ is homeomorphic to the once-punctured torus, T, where the puncture corresponds to the commutator $[A, B] = K^2$. Thus Γ_0 is a *fuchsian*

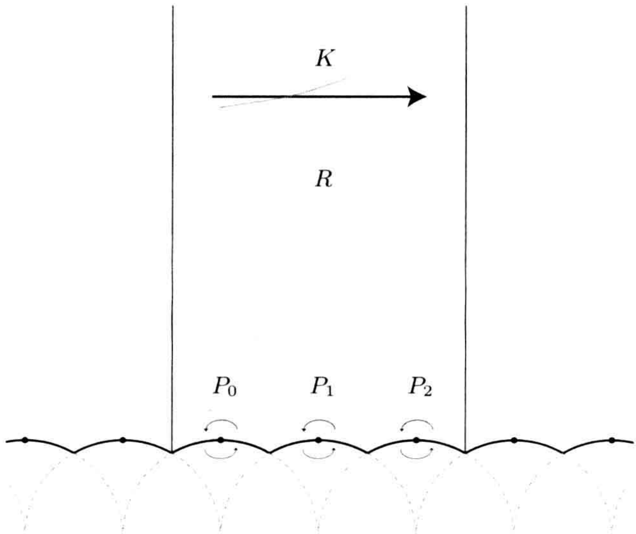

Fig. 0.11. Fuchsian group $\Gamma = \langle P_0, P_1, P_2 \rangle$ and its fundamental region R

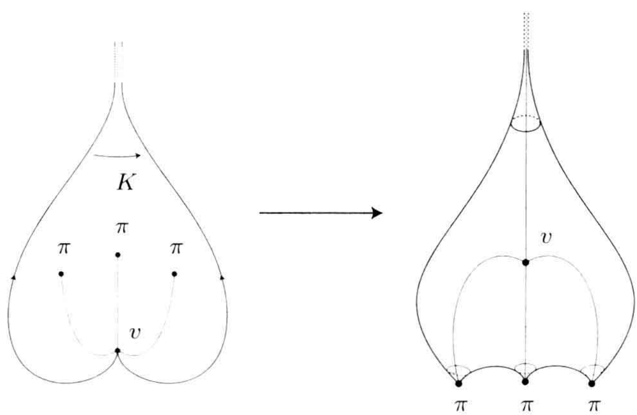

Fig. 0.12. By applying the edge pairings P_0, P_1 and P_2 to the fundamental region R, we obtained the surface on the left hand side. By further applying the edge pairing K to this surface, we obtain the orbifold $\mathcal{O}$ with underlying space once-punctured sphere and with three cone points of cone angle π.

punctured torus group, i.e., it is a discrete free group generated by two elements with parabolic commutator. It is well-known that every fuchsian punctured torus group has a $\mathbb{Z}/2\mathbb{Z}$-extension with quotient homeomorphic to $\mathcal{O}$ as a topological orbifold (cf. [40, Sect. 2]). Thus we abuse terminology and call the extended group a *fuchsian punctured torus group*.

Now look at the region, F, exterior to all L_j. Then this is a "fundamental domain of Γ (resp. Γ_0) modulo $\langle K\rangle$ (resp. $\langle K^2\rangle$)" (cf. Proposition 1.1.3). This region is called the *Ford polygon* of Γ (resp. Γ_0). This can be regarded as the "Dirichlet domain of Γ centered at ∞", because

$$F = \{x \in \mathbb{H}^2 \mid d(x, H_\infty) \le d(x, ZH_\infty) \text{ for every } Z \in \Gamma\},$$

where H_∞ is a sufficiently small horodisk centered at ∞. This implies that the image of ∂F in $\mathbb{H}^2/\Gamma$ is equal to the *cut locus* of $\mathbb{H}^2/\Gamma$ with respect to the cusp, i.e., the set of points of $\mathbb{H}^2/\Gamma$ which has more than two shortest geodesics to the cusp. See Proposition 5.1.3, for a description of the Ford polygons of general fuchsian punctured torus groups.

5.2. 3-dimensional picture of the fuchsian punctured torus group

Figure 0.3 gives a 3-dimensional picture of the group Γ in Fig. 0.11. The elliptic transformation P_j acts on the upper half space model $\mathbb{H}^3$ of the hyperbolic 3-space as the π-rotation around the geodesic joining the two points $(j \pm \mathrm{i})/3$, where $\mathrm{i} = \sqrt{-1}$. (Here we identify the complex plane $\mathbb{C}$ with the boundary of the closure $\overline{\mathbb{H}}^3 = \mathbb{H}^3 \cup \mathbb{C}$.) The *isometric circle*

$$I(P_j) = \{z \in \mathbb{C} \mid |P_j'(z)| = 1\}$$

has center $c(P_j) = j/3$ and radius $1/3$. The hyperplane of $\mathbb{H}^3$ bounded by the isometric circle $I(P_j)$ is called the *isometric hemisphere* of P_j and is denoted by $Ih(P_j)$. Then P_j interchanges the exterior $Eh(P_j)$ and the interior $Dh(P_j)$ of the isometric hemisphere $Ih(P_j)$, and acts on $Ih(P_j)$ as a Euclidean isometry. By the argument in Subsection 5.1, we see that the common exterior $\cap_j Eh(P_j)$, where j runs over $\mathbb{Z}$, is a "fundamental domain of the action of Γ (resp. Γ_0) on $\mathbb{H}^3$ modulo $\langle K\rangle$ (resp. $\langle K^2\rangle$)". Thus it is equal to the *Ford domain* $Ph(\Gamma)$ of Γ, which is defined to be the common exteriors of the isometric hemispheres of all elements of Γ that do not fix ∞ (see Definition 1.1.2 and Proposition 1.1.3). As in the previous subsection, the Ford domain can be regarded as the "Dirichlet domain of Γ centered at ∞", namely

$$Ph(\Gamma) = \{x \in \mathbb{H}^3 \mid d(x, H_\infty) \le d(x, ZH_\infty) \text{ for every } Z \in \Gamma\},$$

where H_∞ is a sufficiently small horoball centered at ∞. Thus the image of $\partial Ph(\Gamma)$ in $\mathbb{H}^3/\Gamma \cong \mathcal{O} \times (-1,1)$ is equal to the *cut locus* of $\mathbb{H}^3/\Gamma$ with respect