PROJECTIVE GEOMETRY

of n dimensions

VOLUME TWO OF INTRODUCTION TO

MODERN ALGEBRA AND MATRIX THEORY

OTTO SCHREIER

EMANUEL SPERNER

CHELSEA PUBLISHING COMPANY

NEW YORK

FIRST ENGLISH-LANGUAGE EDITION

THE PRESENT WORK AND INTRODUCTION TO MODERN ALGEBRA AND MATRIX THEORY TOGETHER CONSTITUTE A COMPLETE AND UNABRIDGED TRANSLATION OF THE GERMAN-LANGUAGE BOOK EINFUEHRUNG IN DIE ANALYTISCHE GEOMETRIE UND ALGEBRA (VOLUMES ONE AND TWO) BY OTTO SCHREIER AND EMANUEL SPERNER. THE PRESENT WORK IS TRANSLATED BY THE LATE PROFESSOR CALVIN A. ROGERS.

LIBRARY OF CONGRESS CATALOG CARD NO. 60-8967

PRINTED IN THE UNITED STATES OF AMERICA

EDITOR'S PREFACE

With the present publication of *Projective Geometry*, the project of translating the famous German-language textbook *Einfuhrung in die analytische Geometrie und Algebra*, by Otto Schreier and Emanuel Sperner, originally published in two volumes, is now complete. As is well known, the purpose of that textbook was to offer a course in Algebra and Analytic Geometry which, when supplemented with a course on the Calculus, would give the student all he needs for a profitable continuation of his studies in modern mathematics. The Preface to the German Edition (see below) gives a more detailed description of the two volumes.

The only change that has been made has been to divide the two volumes somewhat differently in order that they might be usable independently. The first volume and the early part of the second volume were combined into a single book under the title *Introduction to Modern Algebra and Matrix Theory*. The balance, consisting of the major portion of the second volume is published herewith as *Projective Geometry of n Dimensions*. The titles of the two books indicate their respective contents.

The chief prerequisite for reading the present book, aside from a few elementary facts about affine space and systems of linear equations, is a knowledge of the elements of matrix theory such as is contained, for example, in the first four sections of Chapter V (Linear Transformations and Matrices) of *Introduction to Modern Algebra and Matrix Theory*.

Professor Calvin A. Rogers, the translator of the present volume, died before the preparation of the manuscript for the press was begun. The numerous questions that always call for consultation between editor and translator were referred to Professor Abe Shenitzer, whom the Editor wishes to thank for his very considerable help. The Editor also wishes to thank Professor F. Steinhardt. The final form of the manuscript is, of course, the responsibility of the Editor alone.

FROM THE PREFACE TO THE GERMAN EDITION

Otto Schreier had planned, a few years ago, to have his lectures on Analytic Geometry and Algebra published in book form. Death overtook him in Hamburg on June 2, 1929, before he had really begun to carry out his plan. The task of doing this fell on me, his pupil. I had at my disposal some sets of lecture notes taken at Schreier's courses, as well as a detailed (if not quite complete) syllabus of his course drawn up at one time by Otto Schreier himself. Since then, I have also given the course myself, in Hamburg, gaining experience in the process.

In writing this textbook,[1] which is to be published in two volumes, I have followed Schreier's own presentation as closely as possible, so that it might retain the characteristics impressed on the subject matter in Otto Schreier's treatment. In particular, as regards choice and arrangement of material, I have followed Schreier's outline faithfully, except for a few changes of minor importance.

This textbook is motivated by the idea of offering the student, in two basic courses on Calculus and Analytic Geometry, all that he needs for a profitable continuation of his studies in accordance with modern requirements. It is evident that this implies a stronger emphasis than has been customary on algebra, in line with the recent developments in that subject.

The prerequisites for reading this book are few indeed. For the early parts, a knowledge of the real number system—such as is acquired in the first few lectures of almost any calculus course—is sufficient. The later chapters make use of some few theorems on continuity of real functions and on sequences of real numbers. These also will be familiar to the student from the calculus. In some sections which give intuitive interpretations of the subject matter, use is made of some well-known theorems of elementary geometry, whose derivation on an axiomatic basis would of course be beyond the scope of this text.

[1] See the preceding Editor's Preface.

What the book contains may be seen in outline by a glance at the table of contents. The student is urged not to neglect the exercises at the end of each section; among them will be found many an important addition to the material presented in the text.

• • •

The authors' earlier book on matrices has been incorporated into [Chapter V of *Introduction to Modern Algebra and Matrix Theory*],[1] with a few re-arrangements and omissions in order to achieve a more organic whole. The arrangement of material in this chapter is such that the first four sections of the chapter contain essentially all that is needed for [*Projective Geometry*].

• • •

To Mr. W. Blaschke (Hamburg) I owe a debt of gratitude for his continuous interest and help. I also wish to thank Messrs. O. Haupt (Erlangen) and K. Henke (Hamburg) for many valuable hints and suggestions. In preparing the manuscript, I have had the untiring assistance of my wife. For reading the proofs I am indebted to Mr. H. Bückner (Königsberg) in addition to those named above.

Königsberg, October 1935

EMANUEL SPERNER

[1] See the preceding Editor's Preface.

CONTENTS

CHAPTER I

n-DIMENSIONAL PROJECTIVE SPACE

For certain geometrical questions, whose study is central to this book, it is advantageous to extend affine (or euclidean) space by adding to it certain new points, the so-called points at infinity. This procedure is suggested by quite elementary geometrical facts. For example, in order to avoid the oftentimes awkward distinction between intersecting and parallel lines in a plane, we are tempted to ascribe to parallel lines a point of intersection 'at infinity.' Another case in point is afforded by the fact when one line of the affine plane is projected onto another by means of central projection[1] this does not in general establish a one-to-one correspondence between the points of these two lines, whereas it may be made into such a correspondence by an appropriate adjunction of points at infinity. The same is true for the central projection of two planes in space upon each other.

Our immediate task, then, will be to establish and to give a precise analytic description of the introduction of these points at infinity.

Extension of the Affine Plane to the Projective Plane

Because of its intuitive appeal, we shall start with the two-dimensional case.

We shall first of all introduce new coordinates in the affine plane (the so-called homogeneous coordinates). In doing this, we begin with

[1] The central projection upon each other of two lines g and h with respect to a center of projection S is defined by the following rule: P, on g, is taken as the image of Q, on h, and conversely, Q is taken as the image of P, if P, Q, and S lie on a line.

It is therefore clear that P_0 on g has no image point on h if P_0S is parallel to h. Similarly, Q_0 has no image point on g if Q_0S is parallel to g. If we let P_0 correspond to *one* point at infinity on h and Q_0 to *one* point at infinity on g, then exactly one point of h is associated with each point of g, and conversely linear coordinates and hence take as our starting point a *fixed* linear coordinate system in the plane. We get in this way a definite one-to-one correspondence between the points of the plane and the ordered pairs of real numbers. If a point P has the coordinates x_1, x_2, we write $P = (x_1, x_2)$.

Next, we consider all the *ordered triples* of real numbers (ξ_0, ξ_1, ξ_2) for which $\xi_0 \neq 0$. These number triples and the points of the plane are now put into correspondence by means of the following rule:

$P = (x_1, x_2)$ and an ordered triple (ξ_0, ξ_1, ξ_2) with $\xi_0 \neq 0$ are to correspond to each other if and only if:

$$x_1 = \frac{\xi_1}{\xi_0}, \quad x_2 = \frac{\xi_2}{\xi_0}.$$

It follows immediately from this that to each triple (ξ_0, ξ_1, ξ_2) there corresponds only one point, namely, the point with linear coordinates $\frac{\xi_1}{\xi_0}, \frac{\xi_2}{\xi_0}$. On the other hand, to each point $P = (x_1, x_2)$ there correspond infinitely many number-triples. For, the point P obviously corresponds to the triples (ξ_0, ξ_1, ξ_2) and $(\lambda\xi_0, \lambda\xi_1, \lambda\xi_2)$ for arbitrary real $\lambda \neq 0$, since

$$\frac{\xi_i}{\xi_0} = \frac{\lambda\,\xi_i}{\lambda\,\xi_0} \quad (i = 1, 2).$$

Furthermore, the following holds: If two number-triples (ξ_0, ξ_1, ξ_2) and (ξ'_0, ξ'_1, ξ'_2) with $\xi_0, \xi'_0 \neq 0$, correspond to the same point, then there exists a $\lambda \neq 0$ such that $\xi'_i = \lambda\,\xi_i$, $i = 0, 1, 2$. For from $\frac{\xi'_i}{\xi'_0} = \frac{\xi_i}{\xi_0}$ $(i = 1, 2)$ it follows immediately that $\xi'_i = \frac{\xi'_0}{\xi_0}\,\xi_i$. Thus, $\frac{\xi'_0}{\xi_0}$ is the desired λ.

Hence, it is also evident that all the triples corresponding to the same point may be obtained from a given one of them (ξ_0, ξ_1, ξ_2) by multiplying it by an arbitrary real $\lambda \neq 0$.

In particular, all the triples associated with the point $P = (x_1, x_2)$ are of the form $(\lambda, \lambda x_1, \lambda x_2)$, since $(1, x_1, x_2)$ is one particular triple of this kind.

Since the numbers ξ_i of one of our triples (ξ_0, ξ_1, ξ_2) uniquely determine the corresponding point, we may regard them as the coordinates of that point. The coordinates introduced with the help of this correspondence are called *homogeneous coordinates,* or *ratio coordinates* (since they are determined only up to a common constant of proportionality).

Now let $P = (x_1, x_2)$ be a fixed point in the plane, distinct, however, from the origin (Fig. 1). If we now set $Q = (\lambda x_1, \lambda x_2)$ and let λ vary from $+1$ to $+\infty$, then the point Q moves along the line g determined by the points O and P (Fig. 1), from P outward to infinity (in the direction of the arrow).

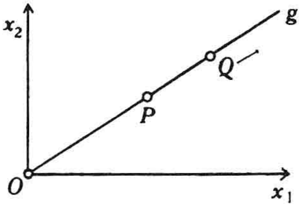

Fig. 1.

We can take $\xi_0 = \frac{1}{\lambda}$, $\xi_1 = x_1, \xi_2 = x_2$ as homogeneous coordinates for Q. Then as $\lambda \to \infty$, we have $\xi_0 \to 0, \xi_1 \to x_1, \xi_2 \to x_2$. We are accordingly led to look upon $0, x_1, x_2$ as the homogeneous coordinates of a point *'at infinity'* (or *improper* point). It is clear that only the ratio of the three coordinates is of significance here, for instead of considering the homogeneous coordinates of Q to be $\frac{1}{\lambda}$, x_1, x_2, we could equally well have thought of them as being $\frac{\varrho}{\lambda}, \varrho x_1, \varrho x_2$, with any fixed ϱ (independent of λ). Upon passing to the limit as $\lambda \to \infty$, we then obtain $0, \varrho x_1, \varrho x_2$ as coordinates of the point at infinity of g.

In all of this, the point $P = (x_1, x_2)$ must be different from the origin. Hence we can ascribe to the triple $(0, 0, 0)$ neither a point in the finite part of the plane nor a point at infinity. For this reason, we once and for all rule out the triple $(0, 0, 0)$; it shall not designate any point whatever.

Finally, then, we have the following definition:

Every triple $(0, \xi_1, \xi_2)$ in which not both ξ_1 and ξ_2 vanish is called a point at infinity or, better, an improper point of the plane. Two improper points $(0, \xi_1, \xi_2)$ and $(0, \xi_1', \xi_2')$ are said to be equal (or coincident) whenever there exists a $\lambda \neq 0$ such that $\xi_1 = \lambda \xi_1'$, $\xi_2 = \lambda \xi_2'$.

The plane obtained by the adjunction of these improper points is called the **projective plane**.

In contradistinction to the improper points, all the points of the projective plane that can be represented by a coordinate triple (ξ_0, ξ_1, ξ_2) with $\xi_0 \neq 0$ are called *proper* points. The totality of proper points is called, as before, the *affine plane*.

We now wish to extend these definitions still further. To each line through the origin of the fixed linear coordinate system we have already assigned a point at infinity. Now, is it desirable to do the same for an arbitrary line, and how can this be accomplished? In order to decide, let us first consider the following question: Can the equation of a line be written in homogeneous coordinates?

In the affine plane a line g can always be represented by an equation of the form

$$a_0 + a_1 x_1 + a_2 x_2 = 0. \tag{1}$$

That is to say, the totality of points whose *linear* coordinates x_1, x_2 satisfy equation (1), fill out a line in the affine plane. Now if P is a proper point of the line g, with linear coordinates x_1, x_2 and homogeneous coordinates ξ_0, ξ_1, ξ_2, then it follows, by the substitution of

$$x_1 = \frac{\xi_1}{\xi_0}, \qquad x_2 = \frac{\xi_2}{\xi_0}$$

into (1), that ξ_0, ξ_1, ξ_2 satisfy the equation

$$a_0 \xi_0 + a_1 \xi_1 + a_2 \xi_2 = 0. \tag{2}$$

And the converse is also true. If ξ_0, ξ_1, ξ_2 satisfy equation (2) and $\xi_0 \neq 0$, then $x_1 = \frac{\xi_1}{\xi_0}$, $x_2 = \frac{\xi_2}{\xi_0}$ satisfy equation (1); that is to say, the triple (ξ_0, ξ_1, ξ_2) represents a point of g.

Thus, we see that equation (2) is satisfied by *all* those triples and *only* those triples (ξ_0, ξ_1, ξ_2), with $\xi_0 \neq 0$, which represent (proper) points of g.[2]

The following definition now suggests itself: *All* those improper points and *only* those improper points whose coordinates satisfy (2) shall belong to g.

How many improper points is that? We claim: Exactly one. For if (ξ_0, ξ_1, ξ_2) is one such point, then the ξ_i must satisfy the following equations:

$$\begin{aligned} a_0 \xi_0 + a_1 \xi_1 + a_2 \xi_2 &= 0, \\ \xi_0 \qquad\qquad &= 0. \end{aligned} \tag{3}$$

This is a system of homogeneous equations in the three unknowns, ξ_0, ξ_1, ξ_2. The rank of the matrix of (3) is 2. For, a_1 and a_2 cannot both

[2] Equation (2) of our line g is homogeneous in the ξ_i, and a similar situation obtains when the equation of any curve is written in terms of these new coordinates; hence the name '*homogeneous coordinates.*'

vanish; else (1) would not represent the equation of a line. According to § 6 of *Modern Algebra*,[3] the totality of the vector solutions $\{\xi_0, \xi_1, \xi_2\}$ of (3) form a one-dimensional linear vector space. That is to say, all the vector solutions are multiples of a *fixed* one among them. This implies, however, that all the triples (ξ_0, ξ_1, ξ_2) that are solutions of (3), with the exception of $(0, 0, 0)$, represent the *same* point in the projective plane (and moreover, by virtue of the second equation of (3), an *improper* point).

We now ask, conversely: Does every homogeneous equation of the form (2) represent a line? Up to now we have seen this to be the case only for such equations of the form (2) as are derivable from an equation of the form (1). In (1), however, a_1 and a_2 must not vanish simultaneously. Let us now consider the case $a_1 = a_2 = 0$. Then (2) reduces to

$$a_0 \xi_0 = 0. \tag{4}$$

If $a_0 = 0$ also, then of course equation (4) no longer represents a line (since (4) is then satisfied by every point of the plane). Thus, let $a_0 \neq 0$. Then (4) is equivalent to:

$$\xi_0 = 0.$$

That is to say: The points that satisfy (4) are precisely all the points at infinity.

Now, for the sake of simplicity, we make the following definition.

The totality of all improper points is called the improper line (or *the line at infinity*).

Thus, we have: *Every homogeneous equation* (2) *in which not all three coefficients vanish, represents a line.*

Now, what can be said about the intersection of two lines conceived of in this extended sense? Let g and h be two lines, g being given by equation (2) and h by

$$b_0 \xi_0 + b_1 \xi_1 + b_2 \xi_2 = 0. \tag{5}$$

The points common to g and h satisfy both equations (2) and (5) and thus are the solutions of the system

[3] *Introduction to Modern Algebra and Matrix Theory*, by O. Schreier and E. Sperner. See Editor's Preface to the present work.

$$(6) \qquad \begin{aligned} a_0 \xi_0 + a_1 \xi_1 + a_2 \xi_2 &= 0, \\ b_0 \xi_0 + b_1 \xi_1 + b_2 \xi_2 &= 0. \end{aligned}$$

The matrix of this system of homogeneous linear equations can have rank 1 or 2.

In the first case, the totality of solutions of (6) is identical with that of each of (2) or (5) separately, that is, the two lines are identical.

In the second case, the vector solutions $\{\xi_0, \xi_1, \xi_2\}$ constitute a one-dimensional linear vector space; that is, there exists exactly one point whose homogeneous coordinates satisfy both the equations (6).

We have thus shown that *any two distinct lines of the projective plane intersect in exactly one point.*

Consequently, parallel lines must also intersect in a point. However, since such lines can have no proper point in common, this point of intersection must be an improper point. From the fact that each line has but one improper point, it follows, in addition, that:

Parallel lines all go through one and the same point at infinity.

On the other hand, non-parallel lines have a finite point of intersection. Their improper points must therefore necessarily be distinct (since two lines have *only one* point of intersection).

In what follows, the definitions that we have adopted for the plane will be generalized to n dimensions ($n > 0$ an arbitrary integer).

n-Dimensional Projective Space

We proceed in complete analogy to the two-dimensional case. We first define homogeneous coordinates in affine R_n by establishing a relation between the points $P = (x_1, x_2, \ldots, x_n)$ of affine R_n and the *ordered* $(n+1)$-tuples $(\xi_0, \xi_1, \ldots, \xi_n)$ of real numbers in which $\xi_0 \neq 0$. This we do in accordance with the following rule.

$P = (x_1, x_2, \ldots, x_n)$ and $(\xi_0, \xi_1, \ldots, \xi_n)$ shall be said to correspond if and only if $x_i = \dfrac{\xi_i}{\xi_0}$ for all $i = 1, 2, \ldots, n$.

According to this rule, we see that, precisely as in the two-dimensional case, just one point of R_n corresponds to each $(n+1)$-tuple $(\xi_0, \xi_1, \ldots, \xi_n)$ with $\xi_0 \neq 0$. Furthermore, two $(n+1)$-tuples $(\xi_0, \xi_1, \ldots, \xi_n)$ and $(\xi'_0, \xi'_1, \ldots, \xi'_n)$ correspond to the same point if and only if there exists a $\lambda \neq 0$ such that $\xi'_0 = \lambda \xi_0, \; \xi'_1 = \lambda \xi_1, \; \cdots, \; \xi'_n = \lambda \xi_n$.

If $P = (x_1, x_2, \ldots, x_n)$ and $(\xi_0, \xi_1, \ldots, \xi_n)$ correspond in accordance with this rule, then the ξ_i are called the *homogeneous coordinates* (or *ratio coordinates*) of P.

The homogeneous coordinates of a point are determined only up to a constant of proportionality; they determine the point, however, uniquely. Let us now adopt the following notation: If a point P has the homogeneous coordinates $\xi_0, \xi_1, \ldots, \xi_n$, we shall write $P = [\xi_0, \xi_1, \ldots, \xi_n]$.[4]

Our final step is the adjunction of the improper points. We adjoin to affine R_n the previously excluded $(n+1)$-tuples $[\xi_0, \xi_1, \ldots, \xi_n]$ in which $\xi_0 = 0$, but in which *not all* the ξ_i vanish simultaneously, and these $(n+1)$-tuples will also be called points; in contradistinction to the '*proper*' points that we have been discussing hitherto, we shall call these new points '*improper*' points (or points '*at infinity*'). Our definition of equality for the improper points (in analogy to that for the proper points) will be as follows: $P = [0, \xi_1, \xi_2, \cdots, \xi_n]$ and $Q = [0, \xi'_1, \xi'_2, \cdots, \xi'_n]$ will be *equal* if and only if there exists a $\lambda \neq 0$ such that $\xi_i = \lambda \xi'_i$ for $i = 1, 2, \ldots, n$.

The extension of affine R_n obtained by adjoining the improper points in this way will be referred to as n-**dimensional projective space** and will be denoted by $\boldsymbol{P}_n$.

We can summarize by saying: Projective P_n consists of the totality of non-trivial[5] ordered $(n+1)$-tuples of real numbers $[\xi_0, \xi_1, \cdots, \xi_n]$, where two such $(n+1)$-tuples $[\xi_0, \xi_1, \cdots, \xi_n]$ and $[\xi'_0, \xi'_1, \cdots, \xi'_n]$ are said to be *equal* (or *coincident*) if and only if there exists a $\lambda \neq 0$ such that $\xi_i = \lambda \xi'_i$ for $i = 0, 1, \ldots, n$.[6]

If $P = [\xi_0, \xi_1, \ldots, \xi_n]$ is a proper point and $x_i = \dfrac{\xi_i}{\xi_0}$ $(i = 1, 2, \ldots, n)$, that is, if $P = (x_1, x_2, \ldots, x_n)$, then we call the x_i the *affine*, or *non-homogeneous*, coordinates of P, in contradistinction to the *homogeneous* coordinates ξ_i.

Now, what is to be understood by a linear subspace in P_n? In affine R_n, a linear subspace of dimension r can always be represented by a system of linear equations

---

[4] We have chosen brackets to avoid confusion with the points of $(n+1)$-dimensional affine space, which we always write in parentheses.

[5] We mean by this the $(n+1)$-tuples in which not all the ξ_i vanish simultaneously. As in the two-dimensional case, we shall once and for all exclude the 'trivial' $(n+1)$-tuple $[0, 0, \ldots, 0]$; it shall not designate any point whatever.

[6] The essential difference between projective P_n and $(n+1)$-dimensional affine R_{n+1} lies in the way in which equality of two points is defined.