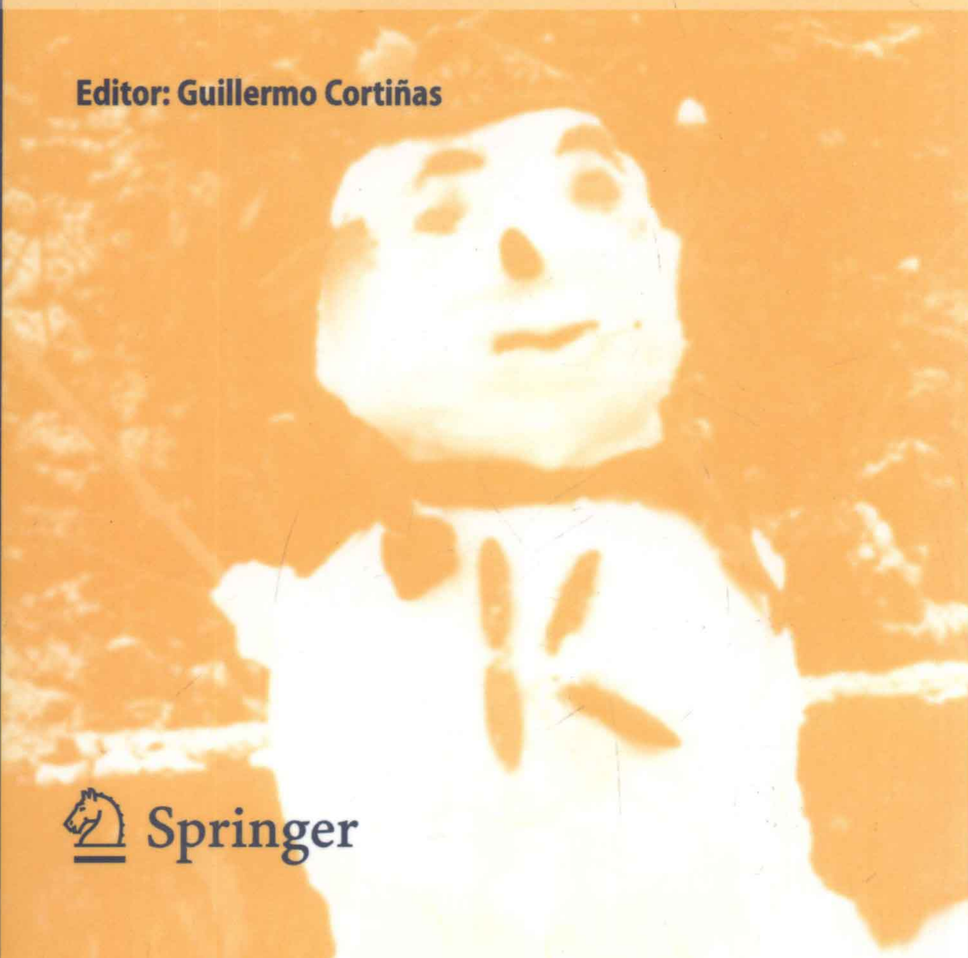
Lecture Notes in Mathematics
2008

Paul Frank Baum
Guillermo Cortiñas
Ralf Meyer
Rubén Sánchez-García
Marco Schlichting
Bertrand Toën

Topics in Algebraic and Topological K-Theory

Editor: Guillermo Cortiñas



Springer

Paul Frank Baum · Guillermo Cortiñas
Ralf Meyer · Rubén Sánchez-García
Marco Schlichting · Bertrand Toën

Topics in Algebraic and Topological K-Theory

Editor: Guillermo Cortiñas

Springer

Paul Frank Baum
The Pennsylvania State University
Mathematics Department
312, McAllister Building
University Park, PA 16802
USA
baum@math.psu.edu

Guillermo Cortiñas
Universidad de Buenos Aires
Facultad de Ciencias Exactas y Naturales
Mathematics Department
C1428EGA Ciudad Autónoma de Buenos Aires
Argentina
gcorti@dm.uba.ar

Ralf Meyer
Universität Göttingen
Mathematisches Institut
Bunsenstr. 3-5
37073 Göttingen
Germany
rameyer@uni-math.gwdg.de

Rubén Sánchez-García
Heinrich-Heine-Universität Düsseldorf
Mathematisches Institut
Universitätsstrasse 1
40225 Düsseldorf
Germany
sanchez@math.uni-duesseldorf.de

Marco Schlichting
University of Warwick
Mathematics Institute
Zeeman Building
Coventry CV4 7AL
United Kingdom
m.schlichting@warwick.ac.uk

Bertrand Toën
Université Montpellier II
Institut de Mathématiques et de Modélisation de Montpellier
Place Eugéne Bataillon
34095 Montpellier
France
btoen@math.univ-montp2.fr

ISBN: 978-3-642-15707-3 e-ISBN: 978-3-642-15708-0
DOI: 10.1007/978-3-642-15708-0
Springer Heidelberg Dordrecht London New York

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2010938612

Mathematics Subject Classification (2010): 19-XX

Cover photo with kind permission by Eugenia Ellis.
It shows a snowman built by the winter school participants.

Cover design: SPi Publisher Services

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Foreword

The five articles of this volume evolved from the lecture notes of Swisk[1], the Sedano Winter School on K-theory held in Sedano, Spain, during the week January 22–27 of 2007. Lectures were delivered by Paul F. Baum, Carlo Mazza, Ralf Meyer, Marco Schlichting, Betrand Toën and myself, for a public of 45 participants. The school was supported by the Ministerio de Educación y Ciencia and the Proyecto Consolider Mathematica of Spain. Funding to cover expenses of US based participants was provided by NSF, through a grant to C.A. Weibel, who was responsible, first of all, for preparing a successful funding application, and then for managing the funds. The local committee, composed of N. Abad and E. Ellis, were in charge of conference logistics. Marco Schlichting collaborated in the Scientific Committee. As organizer of the school and editor of this volume, I am indebted to all these people and institutions for their support, and to my fellow coauthors for their contributions.

Guillermo Cortiñas

[1] The webpage of the school can be found at http://cms.dm.uba.ar/Members/gcorti/workgroup.swisk/index_html.html

Preface

This book evolved from the lecture notes of Swisk, the Sedano Winter School on K-theory held in Sedano, Spain, during the week January 22–27 of 2007. It intends to be an introduction to K-theory, both algebraic and topological, with emphasis on their interconnections. While a wide range of topics is covered, an effort has been made to keep the exposition as elementary and self-contained as possible.

Since its beginning in the celebrated work of Grothendieck on the Riemmann-Roch theorem, applications of K-theory have been found in a variety of subjects, including algebraic geometry, number theory, algebraic and geometric topology, representation theory and geometric and functional analysis. Because of this, mathematicians from each of these areas have become interested in the subject, and they all look at it from their own perspective. On the one hand, this is the richness and appeal of K-theory. On the other hand, it makes it hard to see a global perspective. For example it is not often that an algebraic K-theorist, coming, say, from the algebraic geometry side of the subject, and a topological K-theorist, coming from the functional analysis side, meet together in the same K-theory conference. Thus it is not uncommon to find that algebraic and topological K-theory are regarded as distinct subjects altogether. These notes modestly attempt to illustrate current developments in both branches of the subject, and to emphasize their contacts.

The book is divided into five articles.

The first two are concerned with Kasparov's bivariant K-theory of C^*-algebras and its role in the Baum–Connes conjecture. If G is a locally compact group and A and B are two separable C^*-algebras equipped with a G-action, the Kasparov bivariant K-theory group $KK^G(A,B)$ is defined as the homotopy classes of G-equivariant Hilbert (A,B) bimodules equipped with a suitable Fredholm operator. Kasparov defines an associative product

$$K^G(A,B) \otimes K^G(B,C) \to K^G(A,C)$$

There is an additive category KK^G whose objects are the separable G–C^*-algebras, so that $KK^G(A,B) = \hom_{KK^G}(A,B)$ and composition is given by the Kasparov product. This category is related to usual category G–C^*–Alg of G–C^*-algebras and equivariant $*$-homomorphism by means of a functor $\iota : G\text{–}C^*\text{–Alg} \to KK^G$. The functor ι has the following properties:

- (Stability) If $A \in G\text{–}C^*\text{–Alg}$, and H_1, H_2 are nonzero G-Hilbert spaces, then $\iota(A \otimes K(H_1) \to A \otimes K(H_1 \oplus H_2))$ is an isomorphism.
- (Split Exactness) If $A \xrightarrow{j} B \xrightarrow{p} C$ is a short exact sequence of G–C^*-algebras, split by a G-equivariant homomorphism $s : C \to B$, then $(\iota(j), \iota(s)) : \iota(A) \oplus \iota(C) \to \iota(B)$ is an isomorphism.

Moreover ι is universal (initial) among stable, split exact functors to additive categories. Kasparov theory has many other important properties. To mention one, consider the case when $G = \{1\}$ is the trivial group, and we take $\mathbb{C}$ as the first variable; then

$$K_0(B) = KK(\mathbb{C}, B)$$

is the usual Grothendieck group. Because topological K_1 of a C^*-algebra B is just K_0 of the suspension of B, we also have

$$K_1^{\text{top}}(B) = KK(\mathbb{C}, SB)$$

Thus the whole topological K-theory is recovered from KK, since K^{top} is 2-periodic.

Another application of equivariant KK is in the definition of equivariant K-homology, which plays a fundamental role in the Baum–Connes conjecture. A Hausdorff, locally compact, second countable space X equipped with an action of G by homeomorphisms is called *proper* if the map

$$G \times X \to X \times X, \qquad (g, x) \mapsto (gx, x)$$

is proper, that is, if the inverse image of any compact subspace is compact. A G-subspace $\Delta \subset X$ G is called G-compact if it is proper and the quotient $G \backslash \Delta$ is compact. The equivariant K-homology of a proper G-space X is

$$K_*^G(X) = \underset{\Delta \subset X}{\operatorname{colim}} KK_*^G(C_0(\Delta), \mathbb{C})$$

Here the colimit is taken over all G-compact subspaces $\Delta \subset X$; C_0 is the C^*-algebra of continous functions vanishing at infinity, and $KK_*^G(A, \mathbb{C}) = KK^G(SA, \mathbb{C})$.

The Baum–Connes conjecture proposes a description of the topological K-theory of the reduced C^*-algebra $C_r(G)$ of a locally compact, Hausdorff, second countable group G in terms of G-equivariant K-homology of the universal (final) proper G-space. There is a map

$$K_*^G(\underline{E}G) \to K_*^{\text{top}}(C_r(G))$$

called the assembly map, and the conjecture says it is an isomorphism. The proper G-space $\underline{E}G$ is characterized up to homotopy by the property that any proper G-space X maps to $\underline{E}G$ and that any two such maps are homotopic. The reduced C^*-algebra is defined as follows. If G is a locally compact, second countable group, and μ is a left invariant Haar measure on G, then one can form the separable Hilbert space $H = L^2(G, \mu)$ of square-integrable functions on G. The algebra $C_c(G)$

of compactly supported continuous functions $G \to \mathbb{C}$ with convolution product is faithfully represented inside the algebra $\mathscr{B}(H)$ of bounded operators on H, and $C_r(G)$ is the norm completion of $C_c(G)$.

The conjecture is known to be true for wide classes of groups; no counterexamples are known. There is also a more general version of the conjecture relating the equivarient K-homology of $\underline{E}G$ with coefficients in a separable G–C^*-algebra A with the topological K-theory of the reduced C^*-algebra of G with coefficients in A, $C_r(G,A)$. The latter conjecture is also known for large classes of groups, and is expected to be true in many cases.

The Baum–Connes conjecture is related to a great number of conjectures in functional analysis, algebra, geometry and topology. Most of these conjectures follow from either the injectivity or the surjectivity of the assembly map. A significant example is the Novikov conjecture on the homotopy invariance of higher signatures of closed, connected, oriented, smooth manifolds. This conjecture follows from the injectivity of the rationalized assembly map.

The first article of this volume, K-theory for group algebras, written by P. Baum and R. Sánchez-García, introduces the subject step by step, beginning with the definition of a C^*-algebra, passing through K-theory of C^*-algebras and its connection with Atiyah–Hirzebruch theory, to the general formulation of the Baum–Connes conjecture with coefficients, and of Kasparov's equivariant KK-theory. The latter is introduced in terms of homotopy classes of Hilbert bimodules.

Universal Coefficient Theorems and assembly maps in KK-theory, by R. Meyer, looks at KK-theory and the Baum–Connes conjecture from the point of view of triangulated categories. Equivariant Kasparov theory is introduced using its universal property, and it is explained how this category can be triangulated. The Baum–Connes assembly map is constructed by localising the Kasparov category at a suitable subcategory. Then a general machinery to construct derived functors and spectral sequences in triangulated categories is explained. This produces various generalizations of the Rosenberg–Schochet Universal Coefficient Theorem.

The next article, Algebraic versus topological K-theory: a friendly match, by G. Cortiñas, attempts to be a bridge between the algebraic and topological branches. It presents various variants of algebraic K-theory of rings, including Quillen's, Karoubi–Villamayor's, and Weibel's homotopy algebraic K-theory, denoted respectively K, KV and KH. These variants of algebraic K-theory differ in their behavior with respect to homotopy and excision. Both KV and KH are invariant under polynomial homotopy; if A is any ring, we have $KV_*(A[t]) = KV_*(A)$ and similarly for KH. On the other hand the identity $K_*(A) = K_*(A[t])$ holds in particular cases (e.g. when A is noetherian regular) but not in general. As to excision, if

$$0 \to A \to B \to C \to 0$$

is an exact sequence of (nonunital) rings, then there is a long exact sequence ($n \in \mathbb{Z}$)

$$KH_{n+1}(C) \longrightarrow KH_n(A) \longrightarrow KH_n(B) \longrightarrow KH_n(C)$$

A similar sequence holds for KV under the additional assumption that the sequence be split by a ring homomorphism $C \to B$. The sequence

$$K_{n+1}(C) \to K_n(A) \to K_n(B) \to K_n(C)$$

is exact for $n \leq 0$, but not for $n \geq 1$, in general. Topological K-theory of topological algebras also has several variants, essentially depending on the type of algebras considered. The topological K-theory of Banach algebras is invariant under continuous homotopies; that for locally convex algebras is invariant under C^∞-homotopies. Both satisfy excision and (when suitably stabilized) Bott periodicity: $K_n^{\text{top}} = K_{n+2}^{\text{top}}$.

If A is a topological algebra, there is a comparison map

$$K_*(A) \to K_*^{\text{top}}(A)$$

which is not an isomorphism in general.

Cortiñas' article emphasizes the connections –both formal and concrete– between the algebraic and topological counterparts. For example, Bott periodicity for topological K-theory and the fundamental theorem in algebraic K-theory (which computes the K-groups of the Laurent polynomials) are introduced in a way that makes it clear that each of them is the counterpart of the other. As a concrete connection between algebraic and topological K-theory, the question of whether the comparison map $K_*(A) \to K_*^{\text{top}}(A)$ between the algebraic and topological K-theory of a given topological algebra A is an isomorphism is discussed; Karoubi's conjecture (Suslin–Wodzicki's theorem) establishes that the answer is affirmative for stable C^*-algebras. Proofs of this theorem and of some of its variants are given.

The last two articles approach algebraic K-theory from a categorical point of view.

Higher algebraic K-theory (after Quillen, Thomason and others), by M. Schlichting, introduces higher algebraic K-theory of schemes; emphasis is on the modern point of view where structure theorems on derived categories of sheaves are used to compute higher algebraic K-groups. There are many results in the literature about the structure of triangulated categories, and virtually all of them translate into results about higher algebraic K-groups. The link is provided by an abstract localization theorem due to Thomason and Waldhausen, which –omitting hypothesis– says that a short exact sequence of triangulated categories gives rise to a long exact sequence of algebraic K-groups. This theorem, and its applications, are the heart of the article. Among the main applications presented in the article is Thomason's Mayer–Vietoris theorem, which says that if X is a quasi-compact, quasi-separated scheme and U and V are open quasi-compact subschemes, then there is a long exact sequence

$$K_{n+1}(U \cap V) \to K_n(X) \to K_n(U) \oplus K_n(V) \to K_n(U \cap V) \to K_{n-1}(X)$$

Although the particular case of this result for regular noetherian separated schemes follows from Quillen's early work in the 1970s, the full generality was obtained only twenty years later, by Thomason. The use of derived categories is essential in its proof. Another application is Thomason's blow-up formula. If $Y \subset X$ is a regular embedding of pure codimension d with X quasi-compact and separated, and X' is the blow-up of X along Y, then

$$K_*(X') = K_*(X) \oplus \bigoplus_{i=1}^{d-1} K_*(Y)$$

The methods explained in Schlichting's paper can also be applied to any of the other (co-) homology theories which satisfy an analog of Thomason–Waldhausen's localization theorem; these include Hochschild homology, (negative, periodic, ordinary) cyclic homology, topological Hochschild (and cyclic) homology, triangular Witt groups and higher Grothendieck–Witt groups (the last two when 2 is invertible).

Lectures on dg-categories, by B. Toën, provides an introduction to this theory, which is deeply intertwined with K-theory. The connection comes from the fact that the categories of complexes of sheaves on a scheme are dg-categories. The approach to the subject emphasizes the localization problem, in the sense of category theory. In the same way that the notion of complexes is introduced for the need of derived functors, dg-categories are introduced here for the need of a "derived version" of the localization construction. The existence and properties of this localization are then studied. The notion of triangulated dg-categories, which is a refined version of the usual notion of triangulated categories, is presented, and it is shown that many invariants (such as K-theory, Hochschild homology, . . .) are invariants of dg-categories, though it is known that they are not invariants of triangulated categories. Finally the notion of saturated dg-categories is given and it is explained how they can be used in order to define a secondary K-theory.

June, 2010

Paul F. Baum (University Park)
Guillermo Cortiñas (Buenos Aires)
Ralf Meyer (Göttingen)
Rubén J. Sánchez-García (Düsseldorf)
Marco Schlichting (Coventry)
Betrand Toën (Montpellier)

Contents

K-Theory for Group C^*-algebras

Paul F. Baum[1] and Rubén J. Sánchez-García[2]

[1] Mathematics Department,
206 McAllister Building,
The Pennsylvania State University,
University Park, PA 16802,
USA,
baum@math.psu.edu
[2] Mathematisches Institut,
Heinrich-Heine-Universität Düsseldorf,
Universitätsstr. 1,
40225 Düsseldorf,
Germany,
sanchez@math.uni-duesseldorf.de

1 Introduction

These notes are based on a lecture course given by the first author in the Sedano Winter School on K-theory held in Sedano, Spain, on January 22–27th of 2007. They aim at introducing K-theory of C^*-algebras, equivariant K-homology and KK-theory in the context of the Baum–Connes conjecture.

We start by giving the main definitions, examples and properties of C^*-algebras in Sect. 2. A central construction is the reduced C^*-algebra of a locally compact, Hausdorff, second countable group G. In Sect. 3 we define K-theory for C^*-algebras, state the Bott periodicity theorem and establish the connection with Atiyah–Hirzebruch topological K-theory.

Our main motivation will be to study the K-theory of the reduced C^*-algebra of a group G as above. The Baum–Connes conjecture asserts that these K-theory groups are isomorphic to the equivariant K-homology groups of a certain G-space, by means of the index map. The G-space is the universal example for proper actions of G, written $\underline{E}G$. Hence we proceed by discussing proper actions in Sect. 4 and the universal space $\underline{E}G$ in Sect. 5.

Equivariant K-homology is explained in Sect. 6. This is an equivariant version of the dual of Atiyah–Hirzebruch K-theory. Explicitly, we define the groups $K_j^G(X)$ for $j = 0, 1$ and X a proper G-space with compact, second countable quotient $G\backslash X$. These are quotients of certain equivariant K-cycles by homotopy, although the precise definition of homotopy is postponed. We then address the problem of extending the definition to $\underline{E}G$, whose quotient by the G-action may not be compact.

P.F. Baum et al., *Topics in Algebraic and Topological K-Theory*,
Lecture Notes in Mathematics 2008, DOI 10.1007/978-3-642-15708-0_1,

In Sect. 7 we concentrate on the case when G is a discrete group, and in Sect. 8 on the case G compact. In Sect. 9 we introduce KK-theory for the first time. This theory, due to Kasparov, is a generalization of both K-theory of C^*-algebras and K-homology. Here we define $KK_G^j(A,\mathbb{C})$ for a separable C^*-algebra A and $j = 0, 1$, although we again postpone the exact definition of homotopy. The already defined $K_j^G(X)$ coincides with this group when $A = C_0(X)$.

At this point we introduce a generalization of the conjecture called the Baum–Connes conjecture with coefficients, which consists in adding coefficients in a G-C^*-algebra (Sect. 10). To fully describe the generalized conjecture we need to introduce Hilbert modules and the reduced crossed-product (Sect. 11), and to define KK-theory for pairs of C^*-algebras. This is done in the non-equivariant situation in Sect. 12 and in the equivariant setting in Sect. 13. In addition we give at this point the missing definition of homotopy. Finally, using equivariant KK-theory, we can insert coefficients in equivariant K-homology, and then extend it again to $\underline{E}G$.

The only ingredient of the conjecture not yet accounted for is the index map. It is defined in Sect. 14 via the Kasparov product and descent maps in KK-theory. We finish with a brief exposition of the history of K-theory and a discussion of Karoubi's conjecture, which symbolizes the unity of K-theory, in Sect. 15.

We thank the editor G. Cortiñas for his colossal patience while we were preparing this manuscript, and the referee for her or his detailed scrutiny.

2 C^*-algebras

We start with some definitions and basic properties of C^*-algebras. Good references for C^*-algebra theory are [1, 15, 39] or [41].

2.1 Definitions

Definition 1. *A* Banach algebra *is an (associative, not necessarily unital) algebra A over* $\mathbb{C}$ *with a given norm* $\|\ \|$

$$\|\ \| : A \longrightarrow [0,\infty)$$

such that A is a complete normed algebra, that is, for all $a,b \in A$, $\lambda \in \mathbb{C}$,

(a) $\|\lambda a\| = |\lambda|\|a\|$
(b) $\|a+b\| \leq \|a\| + \|b\|$
(c) $\|a\| = 0 \Leftrightarrow a = 0$
(d) $\|ab\| \leq \|a\|\|b\|$
(e) Every Cauchy sequence is convergent in A (with respect to the metric $d(a,b) = \|a-b\|$*)*

A C^*-algebra is a Banach algebra with an involution satisfying the *C^*-algebra identity*.

Definition 2. *A C^*-algebra $A = (A, \| \ \|, *)$ is a Banach algebra $(A, \| \ \|)$ with a map $* : A \to A$, $a \mapsto a^*$ such that for all $a, b \in A$, $\lambda \in \mathbb{C}$*

(a) $(a+b)^* = a^* + b^*$
(b) $(\lambda a)^* = \overline{\lambda} a^*$
(c) $(ab)^* = b^* a^*$
(d) $(a^*)^* = a$
(e) $\|aa^*\| = \|a\|^2$ *(C^*-algebra identity)*

Note that in particular $\|a\| = \|a^*\|$ for all $a \in A$: for $a = 0$ this is clear; if $a \neq 0$ then $\|a\| \neq 0$ and $\|a\|^2 = \|aa^*\| \leq \|a\| \|a^*\|$ implies $\|a\| \leq \|a^*\|$, and similarly $\|a^*\| \leq \|a\|$.

A C^*-algebra is *unital* if it has a multiplicative unit $1 \in A$. A *sub-C^*-algebra* is a non-empty subset of A which is a C^*-algebra with the operations and norm given on A.

Definition 3. *A* $*$-homomorphism *is an algebra homomorphism $\varphi : A \to B$ such that $\varphi(a^*) = (\varphi(a))^*$, for all $a \in A$.*

Proposition 1. *If $\varphi : A \to B$ is a $*$-homomorphism then $\|\varphi(a)\| \leq \|a\|$ for all $a \in A$. In particular, φ is a (uniformly) continuous map.*

For a proof see, for instance, [41, Thm. 1.5.7].

2.2 Examples

We give three examples of C^*-algebras.

Example 1. Let X be a Hausdorff, locally compact topological space. Let $X^+ = X \cup \{p_\infty\}$ be its one-point compactification. (Recall that X^+ is Hausdorff if and only if X is Hausdorff and locally compact.)

Define the C^*-algebra

$$C_0(X) = \left\{ \alpha : X^+ \to \mathbb{C} \,|\, \alpha \text{ continuous}, \alpha(p_\infty) = 0 \right\},$$

with operations: for all $\alpha, \beta \in C_0(X), p \in X^+, \lambda \in \mathbb{C}$

$$\begin{aligned} (\alpha + \beta)(p) &= \alpha(p) + \beta(p), \\ (\lambda \alpha)(p) &= \lambda \alpha(p), \\ (\alpha\beta)(p) &= \alpha(p)\beta(p), \\ \alpha^*(p) &= \overline{\alpha(p)}, \\ \|\alpha\| &= \sup_{p \in X} |\alpha(p)|. \end{aligned}$$

Note that if X is compact Hausdorff, then

$$C_0(X) = C(X) = \{\alpha : X \to \mathbb{C} \,|\, \alpha \text{ continuous}\}.$$

Example 2. Let H be a Hilbert space. A Hilbert space is *separable* if it admits a countable (or finite) orthonormal basis. (We shall deal with separable Hilbert spaces unless explicit mention is made to the contrary.)

Let $\mathscr{L}(H)$ be the set of bounded linear operators on H, that is, linear maps $T: H \to H$ such that

$$\|T\| = \sup_{\|u\|=1} \|Tu\| < \infty,$$

where $\|u\| = \langle u,u\rangle^{1/2}$. It is a complex algebra with

$$(T+S)u = Tu + Su,$$
$$(\lambda T)u = \lambda(Tu),$$
$$(TS)u = T(Su),$$

for all $T,S \in \mathscr{L}(H)$, $u \in H$, $\lambda \in \mathbb{C}$. The norm is the operator norm $\|T\|$ defined above, and T^* is the adjoint operator of T, that is, the unique bounded operator such that

$$\langle Tu,v\rangle = \langle u,T^*v\rangle$$

for all $u,v \in H$.

Example 3. Let $\mathscr{L}(H)$ be as above. A bounded operator is *compact* if it is a norm limit of operators with finite-dimensional image, that is,

$$\mathscr{K}(H) = \{T \in \mathscr{L}(H)\,|\,T \text{ compact operator}\} = \overline{\{T \in \mathscr{L}(H)\,|\,\dim_{\mathbb{C}} T(H) < \infty\}},$$

where the overline denotes closure with respect to the operator norm. $\mathscr{K}(H)$ is a sub-C^*-algebra of $\mathscr{L}(H)$. Moreover, it is an ideal of $\mathscr{L}(H)$ and, in fact, the only norm-closed ideal except 0 and $\mathscr{L}(H)$.

2.3 The Reduced C^*-algebra of a Group

Let G be a topological group which is locally compact, Hausdorff and second countable (i.e. as a topological space it has a countable basis). There is a C^*-algebra associated to G, called the *reduced C^*-algebra of G*, defined as follows.

Remark 1. We need G to be locally compact and Hausdorff to guarantee the existence of a Haar measure. The countability assumption makes the Hilbert space $L^2(G)$ separable and also avoids some technical difficulties when later defining Kasparov's KK-theory.

Fix a left-invariant Haar measure dg on G. By left-invariant we mean that if $f: G \to \mathbb{C}$ is continuous with compact support then

$$\int_G f(\gamma g)dg = \int_G f(g)dg \qquad \text{for all } \gamma \in G.$$