ADVANCED CALCULUS

by

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PREFACE

This book is intended to provide sufficient material for a course in advanced calculus up to a year in length. It is hoped that the great variety of topics covered will also make this work useful as a reference book.

The background assumed is that usually obtained in freshman and sophomore courses in algebra, analytic geometry, and calculus. The introductory chapter provides a concise review of these subjects; it also serves as a handy reference list of basic definitions and formulas.

The subject matter of the book includes all the topics usually to be found in texts on advanced calculus. However, there is more than the usual emphasis on applications and on physical motivation. Vectors are introduced at the outset and serve at many points to indicate the intrinsic geometrical and physical significance of mathematical relations. Numerical methods of integration and of solving differential equations are emphasized, both because of their practical value and because of the insight they give into the limit process.

A high level of rigor is maintained throughout. Definitions are clearly labeled as such and all important results are formulated as theorems. A few of the finer points concerning the real number system (the Heine-Borel theorem, Weierstrass-Bolzano theorem, and related notions) are omitted. The theorems whose proofs depend on these tools are stated without proof, with references to more advanced treatises. A competent teacher can easily fill in these gaps, if so desired, and thereby present a complete course in real analysis.

A large number of problems, with answers, are distributed throughout the text. They include simple exercises of the "drill" type and more elaborate ones planned to stimulate critical reading. Some of the finer points of the theory are relegated to the problems, with hints given where appropriate.

Generous references to the literature are given and each chapter concludes with a list of books for supplementary reading.

Topical summary. Chapter 1 introduces vectors and their simplest properties, with applications in geometry and mechanics. Partial derivatives are taken up in the second chapter, at first without reference to vectors and then with the aid of vectors for geometric applications. The third chapter introduces the divergence and curl and the basic identities; orthogonal coordinates are treated concisely; the final section concerns n-dimensional vector spaces and is fundamental for the theory of orthogonal functions in Chapter 7.

The fourth chapter, on integration, has as its main goal a clarification of the concept of definite and indefinite integrals. To this end, numerical methods receive special attention. Improper single and multiple integrals are studied and treated in the same way as infinite series, with which they are coordinated at the end of Chapter 6. Chapter 5 is devoted to line and surface integrals. While the notions are first presented without vectors, it very soon becomes clear how natural the vector approach is for this subject. At the end of the chapter an unusually complete treatment of transformation of variables in a multiple integral is given.

Chapter 6 studies infinite series, without assumption of previous knowledge. The notions of upper and lower limits are introduced and used sparingly as a simplifying device; with their aid the theory is given in almost complete form. The usual tests are given, in particular, the root test. With its aid, the treatment of power series is greatly simplified. Uniform convergence is presented with great care and applied to power series. Final sections point out the parallel with improper integrals; in particular, power series are shown to correspond to the Laplace transform.

The seventh chapter is a complete treatment of Fourier series at an elementary level. The first sections give a simple introduction, with many examples; the approach is gradually deepened and a convergence theorem is proved with a minimum of formal work. Orthogonal functions are then studied, with the aid of inner product, norm, and vector procedures. A general theorem on complete systems enables one to deduce completeness of the trigonometric system and Legendre polynomials as a corollary.

Chapter 8 is a fairly concise treatment of differential equations with emphasis on the linear equation and its applications. Problems of forced motion are treated from the "input-output" point of view.

Chapter 9 is exceptionally long and provides a complete, essentially self-contained course in complex variables. The basic integral theorems are deduced as corollaries of Green's theorem. Residues and their applications are treated thoroughly. Conformal mapping is treated at great length, with many examples; this is applied to the Dirichlet problem and to problems in hydrodynamics.

The final chapter, on partial differential equations, lays great stress on the relationship between the problem of forced vibrations of a spring (or a system of springs) and the partial differential equation $\rho u_{tt} + hu_t - k^2\nabla^2 u = F(x, y, z, t)$. By pursuing this idea vigorously the physical meaning of the partial differential equation is made transparent and the mathematical tools used become natural. Numerical methods are also motivated on a physical basis.

Suggestions on the use of this book as the text for a course. It is recommended that the introductory chapter either be omitted from a course outline or else be taken up very briefly. Its main purpose is for reference and as a "refresher" for the student.

The chapters are independent of each other in the sense that each one can be started with a knowledge of only the simplest notions of the previous ones. The later portions of the chapter may depend on some of the later portions of earlier ones. It is thus possible to construct a course using just the earlier portions of several chapters. The following is an illustration of such a plan:

1-1 to 1-14, 2-1 to 2-14, 3-1 to 3-6, 4-1 to 4-4, 4-6 to 4-9, 4-12, 5-1 to 5-6, 6-1 to 6-7, 6-11 to 6-19, 7-1 to 7-5.

It has been found feasible to complete such a program in a one-semester course meeting four hours a week.

If it is desired that one topic be stressed, then the corresponding chapters can be taken up in full detail. For example, Chapters 1, 3, and 5 together provide a very substantial training in vector analysis; Chapters 7 and 10 together contain sufficient material for a one-semester course in partial differential equations; Chapter 9 by itself is a complete elementary course in complex variables.

Sections which are less essential are marked with an asterisk.

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INTRODUCTION

REVIEW OF ALGEBRA, ANALYTIC GEOMETRY AND CALCULUS

In this chapter a review of the fundamentals of algebra, analytic geometry and calculus is presented. The ideas discussed here will serve as the basis of all the theory to follow and reference will often be made to them. Thus this chapter serves both as preparation for the later ones and as a convenient list of formulas and theorems for reference.

0–1 The real number system. The system of real numbers can be thought of as composed of the following:

(a) the *rational numbers:* the positive and negative integers $1, 2, 3, \ldots,$ $-1, -2, -3, \ldots$ and the number 0; the fractions p/q, where p and q are integers;

(b) the *irrational numbers:* numbers expressible as infinite decimals (e.g., $-3.14159\ldots$) but not as ratios of integers.

Together these form a collection of numbers any two of which can be added, subtracted, multiplied, or divided (except for division by zero), subject to the basic rules of algebra:

$$\begin{gathered} a + b = b + a, \quad a \cdot b = b \cdot a, \\ a + (b + c) = (a + b) + c, \quad a \cdot (bc) = (ab) \cdot c, \\ a(b + c) = a \cdot b + a \cdot c, \quad a + 0 = a, \quad a \cdot 1 = a. \end{gathered} \tag{0–1}$$

The real numbers can be identified with the points of an infinite straight line as in Fig. 0–1. On this line a point O has been selected as origin, a unit of length has been chosen and a positive direction assigned. To each number x is then assigned a point P on the line; P is at O if x is 0 and is in the positive or negative direction from O according as x is positive or negative, with the distance OP equal to $|x|$, the absolute value of x (equal to x when x is positive and to $-x$ when x is negative). In this way every number x is represented by one point P and, conversely, each P represents just one number x.

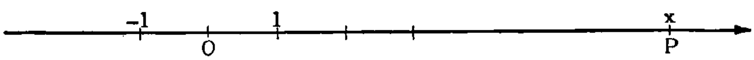

FIG. 0–1. Real numbers.

This geometric picture suggests that the numbers can be ordered: $a > b$ or $b < a$ means simply that $a - b$ is positive or that a lies to the right of b on the above number axis. The $=$ sign, the $>$ or $<$ signs, and the $|\ |$ signs obey the following laws:

Rules of equality:

$$\begin{aligned}&\text{if } a = b \text{ and } b = c, \text{ then } a = c; \quad \text{if } a = b, \text{ then } b = a;\\ &a = a \text{ always}; \quad \text{if } a = a' \text{ and } b = b', \text{ then}\\ &a + b = a' + b' \text{ and } a \cdot b = a' \cdot b'.\end{aligned} \tag{0–2}$$

Rules of inequality:

$$\begin{aligned}&\text{if } a < b \text{ and } b < c, \text{ then } a < c; \quad \text{if } a < b, \text{ then } b > a;\\ &a < a \text{ is impossible}; \quad \text{if } a < a' \text{ and } b < b', \text{ then}\\ &a + b < a' + b' \text{ and, if } a \text{ and } b \text{ are positive, } a \cdot b < a' \cdot b'.\end{aligned} \tag{0–3}$$

Rules for absolute values:

$$\begin{aligned}&|a| \geqq 0; \quad |a| = 0 \text{ if and only if } a = 0;\\ &|a \cdot b| = |a| \cdot |b|; \quad |a + b| \leqq |a| + |b|.\end{aligned} \tag{0–4}$$

0–2 The complex number system. Unless otherwise indicated, the numbers in this text are real. However, the study of the solution of algebraic equations such as

$$x^2 + 1 = 0, \quad x^2 + 2x + 2 = 0$$

leads one to introduce complex numbers of the form $a + bi$, where a and b are real and i, the imaginary unit, has the property: $i^2 = -1$. To each pair of real numbers a, b there corresponds one complex number $a + bi$ and conversely.

The complex numbers can be identified with the points of a plane, the xy plane, in which two perpendicular axes (directed lines) have been chosen and a unit of length has been selected, as in Fig. 0–2. To the complex number $z = x + iy$ corresponds the point P with rectangular coordinates (x, y). The number $0 + 0i = 0$ is represented by the origin O, the point of intersection of the x and y axes.

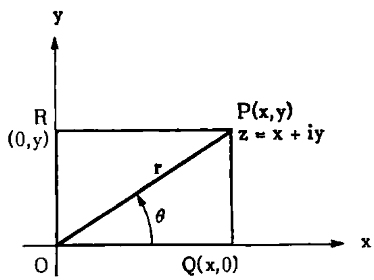

FIG. 0–2. Complex numbers.

The numbers $x + 0i = x$ (*real* numbers) are represented by the points $(x, 0)$ on the x axis just as on the above number axis; the numbers $0 + iy = iy$ (*pure imaginary* numbers) are represented similarly by the points $(0, y)$ on the y axis. The general complex number $z = x + iy$ is represented by the point P whose projection Q on the x axis is $(x, 0)$ and whose projection R on the y axis is $(0, y)$; x is termed the *real part* of z and y the *imaginary part* of z. If $z = x + iy$, then we write: $\bar{z} = x - iy$ and call $\bar{z}$ the *complex conjugate* of z.

The distance $r = OP$ is termed the *absolute value* or *modulus* of the complex number z and is denoted by $|z|$. By the Pythagorean theorem,

$$|z| = r = \sqrt{x^2 + y^2}. \tag{0–5}$$

The signed angle $\theta = \measuredangle XOP$, measured from the positive x axis to OP, is the *argument* or *amplitude* of $x + iy$; angles will be measured positively in the counterclockwise direction and in terms of radians, unless otherwise indicated, so that a complete cycle is 2π. One has then

$$\arg z = \theta = \arcsin \frac{y}{r} = \arccos \frac{x}{r} = \arctan \frac{y}{x}, \tag{0–6}$$

as in trigonometry, θ being determined for given z only up to multiples of 2π. The numbers r, θ are the *polar coordinates* of P.

The complex numbers can be added, subtracted, multiplied, and divided (except for division by zero) and obey the same algebraic rules (0–1) as the real numbers. One has further

$$\begin{aligned}(x_1 + iy_1) + (x_2 + iy_2) &= (x_1 + x_2) + i(y_1 + y_2),\\ (x_1 + iy_1) \cdot (x_2 + iy_2) &= x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1).\end{aligned} \tag{0–7}$$

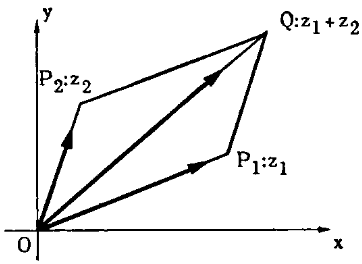

FIG. 0–3. Addition of complex numbers.

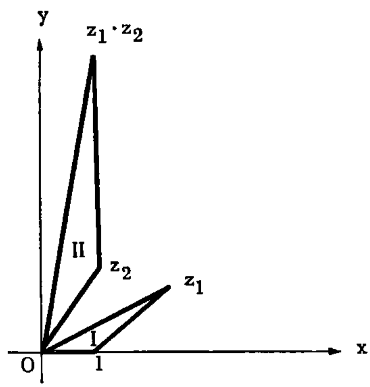

FIG. 0–4. Multiplication of complex numbers.

The first part of (0–7) shows that addition of complex numbers is in accordance with the "parallelogram law" by which forces are added in mechanics, as illustrated in Fig. 0–3. The second part of (0–7), when written in polar coordinates, is the basis of a graphical construction of the product of two complex numbers:

$$\begin{aligned}z_1 \cdot z_2 &= (r_1 \cos\theta_1 + ir_1 \sin\theta_1) \cdot (r_2 \cos\theta_2 + ir_2 \sin\theta_2)\\ &= r_1r_2[\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)]\\ &= r_1r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)].\end{aligned} \tag{0–8}$$

Accordingly,

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|, \quad \arg(z_1 \cdot z_2) = \arg z_1 + \arg z_2; \tag{0–9}$$

from this it follows that the triangles I and II of Fig. 0–4 are similar. Thus $z_1 \cdot z_2$ can be constructed graphically from z_1 and z_2.

The rules of equality for complex numbers are the same as for real numbers, but inequalities between complex numbers have no meaning. The rules for absolute values are the same as for real numbers. The rule

$$|z_1 + z_2| \leqq |z_1| + |z_2| \tag{0-10}$$

expresses the geometric condition: $OQ \leqq OP_1 + P_1Q$ of Fig. 0-3.

0-3 Algebra of real and complex numbers. If x is a real number and n is a positive integer, then x^n, the nth power of x, is defined as $x \cdot x \cdots x$ (n factors). One then verifies the rules

$$x^m \cdot x^n = x^{m+n}, \quad (x^m)^n = x^{mn}. \tag{0-11}$$

A polynomial in x of degree n is an expression of form

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

where $a_0, a_1, \ldots, a_n$ are real numbers and $a_0 \neq 0$. Thus $x^2 + 2x - 3$ is a polynomial in x of degree 2.

An algebraic equation in x of degree n is a polynomial in x of degree n set equal to 0, e.g.,

$$x^2 - 4x - 5 = 0.$$

It is shown in algebra that such an equation has at most n real roots. In particular, if $a > 0$, the equation

$$x^n = a$$

has precisely one positive real root, denoted by $\sqrt[n]{a}$ or by $a^{\frac{1}{n}}$.

The preceding definitions extend at once to powers, polynomials, and algebraic equations involving complex numbers. Thus

$$z^2 + 1 = 0$$

is an algebraic equation of degree 2; so also is

$$(1 - i)z^2 + iz - 1 = 0,$$

in which the coefficients are complex. It is shown in advanced mathematics that an equation of degree n has n complex roots, some of which may coincide; if the coefficients are real, then the imaginary roots come in conjugate pairs.

A linear equation is one of first degree, e.g.,

$$3x - 2 = 0, \quad 5z + 4i = 0.$$

This has always just one solution: $ax + b = 0$ has the root $x = -(b/a)$.

A quadratic equation is one of second degree, e.g.,

$$x^2 - 5x + 6 = 0.$$

The general equation

$$az^2 + bz + c = 0$$

has the roots

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{0-12}$$

If a, b, c are real, then the character of the roots is determined by the *discriminant:* $b^2 - 4ac$. If $b^2 - 4ac > 0$, then the roots are real and distinct; if $b^2 - 4ac = 0$, then the roots are real and equal; if $b^2 - 4ac < 0$, the roots are the conjugate complex numbers $x \pm iy$.

Explicit formulas are available for solving the general third and fourth degree equations. [See L. E. Dickson, *First Course in the Theory of Equations* (New York: Wiley, 1922) Chapter IV.] For equations of higher degree there are no explicit formulas. Numerical and mechanical methods are available for finding real and complex roots of equations of high degree; these are discussed in Chapter VIII of the book by Dickson.

Equations of form

$$z^n = a$$

can be solved for complex z, with a real or complex. The roots are the "nth roots of a." The solution is based on the De Moivre formula

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \tag{0-13}$$

which follows from the rule (0–8) for multiplication. From this one concludes that, if a has polar coordinates r, θ, so that $a = r(\cos\theta + i\sin\theta)$, then

$$\sqrt[n]{a} = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right)\right], \tag{0-14}$$

where k ranges over the values $0, 1, 2, \ldots, n-1$ and $\sqrt[n]{r}$ is the positive real nth root of r. This is illustrated in Fig. 0–5, with $n = 5$.

A system of *simultaneous linear equations* is a system such as

$$a_1x + b_1y + c_1z = k_1, \quad a_2x + b_2y + c_2z = k_2, \quad a_3x + b_3y + c_3z = k_3. \tag{0-15}$$

Here there are three equations in the three unknowns x, y, z; in general one may have n equations in m unknowns. These can in general be solved by combining equations to successively eliminate variables.

For n equations in n unknowns the solution can be expressed in terms of determinants, the theory of which is discussed below. One denotes by D the "determinant of the coefficients"; for (0–15) this is the determinant

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \tag{0-16}$$

In general D is an nth order determinant. One denotes by $D_1, D_2, \ldots$

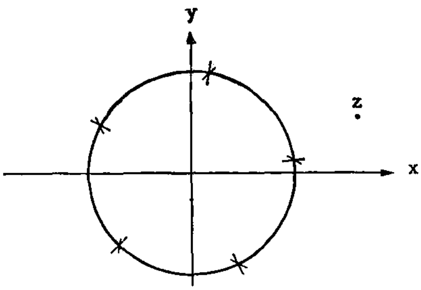

FIG. 0–5. Fifth roots of $z = 2 + i$.

the determinants obtained from D by replacing the first, second, . . . column of D by the numbers $k_1, k_2, \ldots, k_n$. Thus for (0–15)

$$D_2 = \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}. \tag{0–17}$$

The solution of the system of equations is then given by

$$x = \frac{D_1}{D}, \quad y = \frac{D_2}{D}, \quad \ldots; \tag{0–18}$$

this is Cramer's Rule. If $D \neq 0$, (0–18) gives the one and only solution. If $k_1 = k_2 = k_3 = 0$ (*homogeneous case*) and $D \neq 0$, (0–18) gives the trivial solution $x = y = z = 0$; if $D = 0$, there are infinitely many solutions.

For n equations in m unknowns, with $n < m$, one can try to solve as above for n unknowns in terms of the remaining $m - n$ unknowns. This will be possible, provided the corresponding determinant of coefficients D for these n unknowns is not 0. For $n > m$, there are more equations than unknowns; the equations are contradictory unless certain determinants are 0. For a full discussion the reader is referred to Chapter VIII of the book by Dickson cited above.

For linear equations such as (0–15) the coefficients and unknowns may be real or complex numbers; the form of the solutions is the same in either case.

The properties of determinants needed in this book are listed here:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc; \tag{0–19}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}; \tag{0–20}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} + \cdots. \tag{0–21}$$

Thus, in general, a determinant of nth order is defined in terms of determinants of $(n - 1)$st order; the coefficients of $a_1, b_1, \ldots$ on the right side of (0–20) and (0–21) are the "cofactors" of these elements.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}; \tag{0–22}$$

in general, rows and columns can be interchanged.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}; \tag{0–23}$$

in general, interchanging two rows (or columns) multiplies the determinant by -1.

$$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}; \tag{0-24}$$

a factor of any row (or column) can be placed before the determinant.

$$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0; \tag{0-25}$$

if two rows (or columns) are proportional, the determinant equals 0.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} A_1 & b_1 & c_1 \\ A_2 & b_2 & c_2 \\ A_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + A_1 & b_1 & c_1 \\ a_2 + A_2 & b_2 & c_2 \\ a_3 + A_3 & b_3 & c_3 \end{vmatrix}; \tag{0-26}$$

this indicates how two determinants differing only in one row or column can be added.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + ka_2 & b_1 + kb_2 & c_1 + kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}; \tag{0-27}$$

the value of the determinant is unchanged if the elements of one row are multiplied by the same quantity k and added to the corresponding elements of another row. By suitable choice of k, one can use this rule to introduce zeros; by repetition of the process, one can reduce all elements but one in a chosen row to zero. This procedure is basic for numerical evaluation of determinants.

Proofs of these rules and further properties are given in Chapter VIII of the book by Dickson mentioned above.

Three other formulas of algebra will be useful:

$$a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d] = n\frac{a + [a + (n - 1)d]}{2}; \tag{0-28}$$

$$a + ar + ar^2 + \cdots + ar^{n-1} = a\frac{1 - r^n}{1 - r} \quad (r \neq 1); \tag{0-29}$$

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n - 1)}{2!}a^{n-2}b^2 + \cdots + C_r^n a^{n-r}b^r + \cdots + b^n, \tag{0-30}$$

where

$$C_r^n = \frac{n(n - 1)\cdots(n - r + 1)}{r!} = \frac{n!}{r!(n - r)!}. \tag{0-31}$$

These give respectively the sum of an *arithmetic progression*, the sum of a *geometric progression*, and the *binomial theorem*. Throughout n is a positive integer and $n!$ (read "n factorial") is defined as follows: