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Ludger Kaup · Burchard Kaup

Holomorphic Functions of Several Variables

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An Introduction to the Fundamental Theory

With the Assistance of Gottfried Barthel

Translated by Michael Bridgland

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Authors

Dr. Ludger Kaup
Professor of Mathematics
Universität Konstanz

Dr. Burchard Kaup
Professor of Mathematics
Universität Fribourg

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Foreword

χαλεπὰ τὰ καλά

In the preface to his pioneering book, Die Idee der Riemannschen Fläche [Wl], Hermann Weyl wrote

"Erst Klein[1] hat ja jene freiere Auffassung der Riemannschen Fläche recht zur Geltung gebracht, welche ihre Verbindung mit der komplexen Ebene als eine über der Ebene sich ausbreitende Überlagerungsfläche aufhebt, und hatte dadurch den Grundgedanken Riemanns erst seine volle Wirkungskraft gegeben. ... Ich teilte seine Überzeugung, daß die Riemannsche Fläche nicht bloß ein Mittel zur Veranschaulichung der Vieldeutigkeit analytischer Funktionen ist, sondern ein unentbehrlicher sachlicher Bestandteil der Theorie; nicht etwas, was nachträglich mehr oder minder künstlich aus den Funktionen herausdestilliert wird, sondern ihr prius, der Mutterboden, auf dem die Funktionen erst wachsen und gedeihen können."[2]

Likewise in the theory of holomorphic functions of several complex variables, the investigation of holomorphic functions and their natural domains of existence is rooted in an abstract version of a Riemann surface, called a "complex space".

Two phenomena appearing only in the multidimensional theory are particularly striking:

i) For $n > 1$, a domain in $\mathbb{C}^n$ is not in general a *domain of holomorphy*; that is, it need not be the maximal domain of definition of a holomorphic function. Complements of finite point-sets in $\mathbb{C}^n$ fall into that category.

ii) In the generalization of concrete Riemann surfaces to higher dimensions, called "Riemann domains", ramification points need not possess local uniformizations, as can be seen even in such a simple example as the "origin" in the Riemann domain of $\sqrt{z_1 z_2}$. Such points are called *singularities.*

Those two phenomena profoundly influenced the development of (multidimensional) complex analysis; they also determine the content and organization of our book:

i) In the study of domains of holomorphy in $\mathbb{C}^n$ and their function theory, it soon became evident that the concept of "*analytic convexity*" is of fundamental importance. In attempting to free oneself, in the sense indicated by Weyl, of the concrete realization of domains of holomorphy as subsets of $\mathbb{C}^n$, one is led naturally to the concept of

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[1] Felix Klein (1849–1925).

[2] Klein had been the first to develop the freer conception of a Riemann surface, in which the surface is no longer a covering of the complex plane; thereby he endowed Riemann's basic ideas with their full power. ... I shared his conviction that Riemann surfaces are not merely a device for visualizing the many-valuedness of analytic functions, but rather an indispensable essential component of the theory; not a supplement, more or less artificially distilled from the functions, but their native land, the only soil in which the functions grow and thrive. [Wl]$_2$

Stein spaces. Due to their rich function-theoretic structure, such spaces also are called "holomorphically complete"; they play a leading role in complex analysis. Both Chapter 1 in the first part of the book and the entire third part are devoted to the topics just outlined.

ii) Even at its singular points, a Riemann domain can be described locally as an "*analytic set*", i.e., the solution set of a system of holomorphic equations. Also the systematic study of complex manifolds depends heavily on analytic sets, for example, in many inductive arguments. Consequently, function theory on analytic sets is indispensible for an understanding of Riemann domains. A further attempt to obtain a "freer conception" à la Weyl, this time to describe analytic sets without reference to an embedding in a complex number space, leads to the construction of the complex spaces mentioned previously. That explains why they have become the central object of investigation in complex analysis – or, as some authors say in analogy to algebraic geometry, in complex analytic geometry.

A systematic treatment of complex spaces entails the analysis of *punctual*, *local*, and *global* properties. With its investigation of power series algebras and their homomorphic images, Chapter 2 can be viewed as a central portion of the *punctual* theory. The "*Weierstrass Preparation Theorem*" occupies such a key position in that investigation that Chapter 2 might well have been entitled, "Variations on the Weierstrass Preparation Theorem".

In Chapter 3, we prepare for the transition to the local and global theories of complex spaces by developing a more general formal framework in the concept of a "ringed space", which is a topological space with a distinguished "structure sheaf". We then develop a geometric intuition through the presentation of numerous examples of manifolds and reduced complex spaces.

In Chapter 4, the *local* theory is the main topic. The concept of *coherent sheaves* plays a decisive role in the step from "punctual" to "local"; the central results, aside from the actual Coherence Theorems, are the Representation Theorem for Prime Germs 46.1 and the Local Characterization Theorem for Finite Morphisms 45.4.

The *global* theory is particularly well developed for two classes of complex spaces that can be characterized in terms of topological or function-theoretic "completeness", namely, compact spaces and Stein spaces. In this book, the emphasis in the global theory is on Stein spaces, since their function theory yields fundamental results for the local investigation of complex spaces as well. For the step from "local" to "global", we apply cohomology theory; the central result is *Theorem B*: "Stein spaces have trivial analytic cohomology".

In a supplement, Chapter 7, we treat a class of complex spaces whose function theory closely resembles that of manifolds, namely, the "*normal spaces*", characterized by the validity of the Riemann Removable Singularity Theorems. In the Normalization Theorem, we show how to modify an arbitrary reduced complex space so that it becomes a normal space (that may be viewed as a first step toward "regularizing" it into a manifold). A much deeper result is Hironaka's Theorem on the resolution of singularities [Hr], whose proof lies well beyond the scope of this book; however, we do present the tools for the resolution of singularities of complex curves and surfaces: normalization and quadratic transformation.

The coherence of the structure sheaves of complex spaces leads to an interplay between algebra and geometry that we find especially charming, and we have used it as a methodical leitmotiv for Chapter 4. The algebraic objects "*analytic algebras*" (which appear punctually as stalks of structure sheaves) correspond to the local geometric objects "*germs of complex spaces*". As a result, geometric statements have algebraic proofs (which are frequently easier and more transparent), and algebraic statements can be interpreted geometrically. In order to exploit this "antiequivalence" to the fullest, it is necessary to drop the restrictive condition "reduced" from the definition of complex spaces in Chapter 3. Consequently, stalks of structure sheaves of complex spaces, which are simply analytic algebras, may contain *nilpotent elements*. We thereby have an appropriate tool for the treatment of solution sets of systems of holomorphic equations with "multiplicities". For example, consider the zero of the function z^m; its multiplicity is reflected in the analytic algebra $\mathbb{C}\{z\}/(z^m)$, rather than in the corresponding reduced algebra $\mathbb{C}\{z\}/(z)$. That more general approach does not complicate the proofs of the essential results; on the contrary, it is advantageous – for example, for the proof of Cartan's Coherence Theorem. There is a similar antiequivalence, which we discuss to a limited extent in connection with the Character Theorem (§ 57), between Stein spaces on the geometric side and their algebras of global holomorphic functions on the algebraic side.

This introduction to complex analysis is intended both as a textbook and as a guide for independent study. In general, we neither discuss the historical development of the theory, nor mention the discoverers of propositions or proofs. However, those familiar with the subject cannot fail to notice the tremendous influence on our presentation, both direct and indirect, of the Münster school that grew up around Heinrich Behnke (1898–1979), particularly through the ideas of Grauert and Remmert.

The limited scope of an introductory textbook permits the presentation of only a small selection of topics from a rich and living branch of mathematics; the resulting omissions are all the more apparent if the discipline under consideration distinguishes itself through numerous connections to other subjects, as is the case with complex analysis and, say, commutative algebra, differential geometry, algebraic geometry, functional analysis, the theory of partial differential equations, and algebraic topology. The reader who wishes to pursue the subject in greater depth should consult the relevant monographs (such as [Fi] and the appendices in [BeTh] for an overview, [GrRe] and [Ab] for the punctual theory, $[\text{GrRe}]_2$ and [BaSt] for the global theory, [Hö] for $\bar{\partial}$-theory, etc.).

Since this book is an introduction to the foundations of the theory, we generally do not treat subjects that have not already appeared in one form or another in textbooks (two major exceptions are the investigation of quotient structures and part of the discussion of normalization). We have given considerable attention to the (generally simple) exercises, which serve to test the reader's understanding of the material, to vary or complete simple proofs, and sometimes to augment the text. In general, an exercise whose conclusion is needed later in the book is provided with hints, if necessary, and an indication of the place at which it is applied for the first time.

We use small print to indicate both material from other disciplines and supplementary material that may be left out during a first reading of the text without sacrificing rigor.

Aside from sheaf theory and cohomology theory (for which we summarize the necessary concepts and results in such a way as to motivate their application), we assume very little prior knowledge of the reader: some elements of one-dimensional complex analysis, and some basic results from differential calculus, algebra, and functional analysis. Even for those topics, we frequently give explicit references.

We are indebted to many colleagues for valuable suggestions and detailed comments on the text; in particular, we should like to mention Milos Dostal, George Elenczwajg, Gerd Fischer, Wilhelm Kaup, Leopoldo Nachbin, and Reinhold Remmert. We are pleased to acknowledge the work of Karl-Heinz Fieseler and Ernst-Ulrich Kolle, who read the German manuscript with great care, and corrected many inaccuracies. We are also grateful to our students for their contributions, in the form of questions and comments, to the betterment of the text. Mrs. Gisela Schroff patiently and carefully typed the various drafts of both the German and English manuscripts, for which we are particularly grateful. Most especially, our thanks go to Michael Bridgland, who, under circumstances that were not always of the best, has rendered the German manuscript into English with so much care. Finally, we would like to thank Heinz Bauer for suggesting to us that we write this book. The publishers have earned our gratitude both for their patience and understanding during the many delays that accompanied the writing of the book, and for their friendly cooperation during the printing. We also thank everyone else, colleagues and institutions, who have contributed either directly or indirectly to the completion of the book, not least of all our families.

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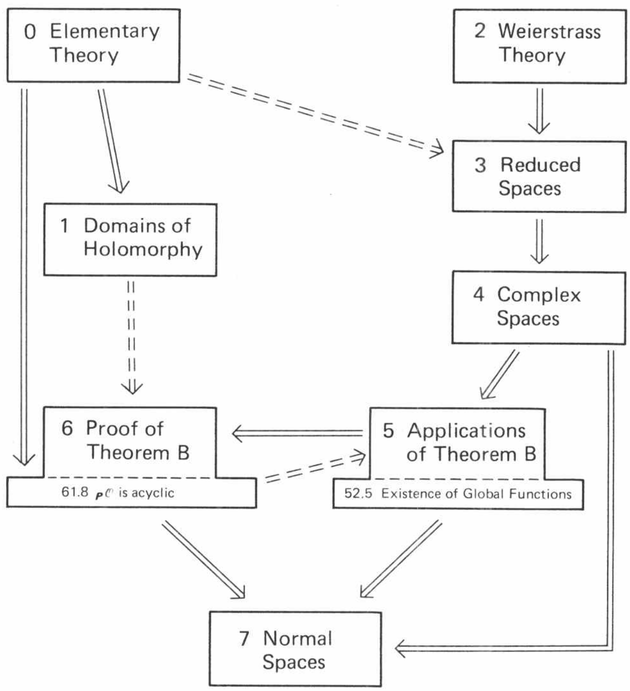
Ludger Kaup
Burchard Kaup

Interdependence of chapters



Courses and Seminars: Guide to the Essentials of Specific Topics

Here we propose ways of using the book for various courses or seminars:

1) *Function theory on domains in* $\mathbb{C}^n$: §§ 1–14.
2) *Complex manifolds*: §§ 1, 3, 4, 6.1–6.6, 8; definition of manifolds with E. 32a; § 32 without the results on singular spaces; § 32 A, B.
3) *Analytic algebras and their dimension theory*: §§ 21–24; 45.6 proved as in [GrRe II.2 Satz 2]; all of the algebraic results of § 48, up to and including 48.11_{alg}, with proofs; possibly E.48e, (32.11.1)–32.13, 45.7, and 45.8, as well; for a geometric interpretation of algebraic statements, we recommend the definition of germs of complex spaces and their morphisms as geometric realizations of analytic algebras and their homomorphisms, to be followed with 44.7–44.14 and 45.1–45.14, possibly without the proof of 45.2.
4) The *theory of complex spaces* can be appended to 3): central themes: Coherence Theorems (Oka, Cartan, Finite), Representation Theorem for Prime Germs, decomposition into irreducible components – § 31, examples from § 32, § 33, and the remainder from Chapter 4 (without the supplement).
5) *Domains of holomorphy*, either in $\mathbb{C}^n$ or in Stein manifolds X: the goal is either 63.7 or 63.2: Chapter 1; §§ 30, 41, 42; specialization to domains in $\mathbb{C}^n$ (resp., manifolds) of §§ 50–53, § 55, 56.2, and Chapter 6. (The Finite Coherence Theorem in the proof of 63.3 can be avoided for manifolds X that admit local coordinates by global functions: then, for each $K = \bar{K} = \hat{K} \subset U \Subset X$, there exists a $\varphi \in \mathrm{Hol}(X, \mathbb{C}^m)$ such that
$$\varphi(K) \subset \boldsymbol{P} := \boldsymbol{P}^m(1),\quad \varphi(\partial U) \cap \boldsymbol{P} = \emptyset,$$
and such that $\varphi|_U$ is an injective mapping that induces on every stalk a surjective homomorphism; hence, $\varphi\colon W := \varphi^{-1}(\boldsymbol{P}) \cap U \to \boldsymbol{P}$ is a (closed) embedding, i.e., W is a B-space and Runge in X, thus in every analytic polyhedron in X that includes W.)
6) *Solution of Levi's problem*: Chapter 1, §§ 61, 62, 63 A (prerequisites: Approximation Theorem, sheaf theory, cohomology theory).
7) *Quotients of complex spaces*: 32 B.3 and its applications in § 32 B, § 49 A (prerequisites: ringed spaces, coherent sheaves, Finite Coherence Theorem).
8) *Normalization*: §§ 45 B, 46 B, 71, 72, possibly 74.
9) *Finiteness Theorem of Cartan-Serre*: §§ 30, 41, 42, 50, 51, 52, and 55; 56.2; §§ 61, 62.
10) *Finite Mappings*: 33.1, §§ 45, 45 B, 46.

Contents

Part Two: Function Theory on Analytic Sets

Part Three: Function Theory on Stein Spaces

Part One: Function Theory on Domains in $\mathbb{C}^n$

We begin our exposition of complex analysis by adapting familiar propositions and methods of proof from the one-dimensional theory to the multidimensional case in Chapter 0. In order to make the analogy between the two cases particularly transparent, we introduce holomorphic functions as continuous partially holomorphic functions; we then show that approach to be equivalent to the other usual definitions.

In Chapter 1, we work out basic differences between the two theories, the existence of which is discernible in Chapter 0 only in connection with the Kugelsatz. Thus, we show that, contrary to the impression left by Chapter 0, the multidimensional theory does not consist of mere adaptations, but rather that completely new methods must be developed.

Chapter 0: Elementary Properties of Holomorphic Functions

In analogy to the historical development of multidimensional function theory, we begin with the basic concepts and results that carry over from the one-dimensional theory relatively easily.

The set $\mathcal{O}(X)$ of holomorphic functions on an open set[1] $X \subset\subset \mathbb{C}^n$ is an algebra that includes the polynomial algebra $\mathbb{C}[Z_1, \ldots, Z_n]$; in the metric topology of compact convergence, it is complete, and Montel's Theorem holds. From Cauchy's Integral Formula, it follows that holomorphic functions are precisely the ones that are represented locally by their Taylor series. Multidimensional versions of the Identity Theorem and the Maximum Principle hold; in particular, every nonconstant holomorphic function is an open mapping. In addition, standard theorems from real analysis, such as the Implicit Function Theorem, the Inverse Function Theorem, and the Rank Theorem, extend to the complex case. In contrast to real differential calculus, a holomorphic injection between equidimensional domains is "biholomorphic" (8.5).

In the multidimensional version of the classical Removable Singularity Theorem, the exceptional sets are not merely isolated points, but rather arbitrary analytic subsets, in other words, solution sets of systems of holomorphic equations; that leads us to investigate the elementary properties of such sets. The First Riemann Removable Singularity Theorem has an interesting extension that is vacuous in the one-dimensional case: if a function is holomorphic outside an exceptional set of codimension at least two, then it is holomorphically extendible (Second Riemann Removable Sin-

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[1] For an open subset A of B, we use the notation $A \subset\subset B$.

gularity Theorem). The proof is based on Hartogs's *Kontinuitätssatz*, which we analyze more carefully in Chapter 1, where we are concerned with fundamental differences between the one- and multidimensional theories.

§ 1 Definition of Holomorphic Functions

Throughout this introductory chapter, X denotes an open subset, called a *region*, of the *complex number space* $\mathbb{C}^n$. If X is also connected, then X is called a *domain*; in that case we often use the letter G (for *Gebiet*). We usually use $z = (z_1, \dots, z_n)$ and $z_j = x_j + iy_j$ to denote the coordinates of $\mathbb{C}^n$ and the decomposition of z_j into real and imaginary parts, respectively. We frequently use the *Euclidean norm* $\|z\|$ and the *maximum norm* $|z|$ defined as follows:

$$\|z\|^2 := \Sigma z_j \bar{z}_j \quad \text{and} \quad |z| := \max_{1 \leq j \leq n} |z_j|.$$

We recall the one-dimensional theory (i.e., $X \subset \mathbb{C}^1$): a continuous function $f: X \to \mathbb{C}$ is called *holomorphic* if it satisfies the following equivalent conditions:

i) f is complex differentiable: $\dfrac{df}{dz}$ exists and coincides with

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i} \cdot \frac{\partial f}{\partial y}\right).$$

ii) Locally, f is representable by a convergent power series.

iii) For each contractible piecewise-smooth closed curve γ in X, $\int_\gamma f(z)dz = 0$ (*Cauchy's Theorem*).

iv) f has continuous partial derivatives with respect to x and y at each point in X, and they satisfy the Cauchy-Riemann differential equation,

$$\frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i} \cdot \frac{\partial f}{\partial y}\right) = 0.$$

With that background, we can generalize to the n-dimensional case.

1.1 Definition. *A function $f: X \to \mathbb{C}$ is called partially holomorphic if, for each fixed $(z_1^0, \dots, z_n^0) \in X$, and each $j = 1, \dots, n$, the function of one variable determined by the assignment*

$$z_j \mapsto f(z_1^0, \dots, z_{j-1}^0, z_j, z_{j+1}^0, \dots, z_n^0)$$

is holomorphic. A continuous partially holomorphic function is called holomorphic, and the set of holomorphic functions on X is denoted by $\mathcal{O}(X)$.

In the one-dimensional case, sums, products, and complex multiples of holomorphic functions are holomorphic; $\mathcal{O}(X)$ is thus an algebra whose unit element is the constant function with value 1. The units of $\mathcal{O}(X)$ are the holomorphic functions with no zeros.

We call a complex vector space A an *algebra*[2] if there is a multiplication on A,

$$A \times A \to A, \quad (a,b) \mapsto a \cdot b,$$

that, with vector space addition, makes A a commutative ring with a unit element such that ring multiplication and scalar multiplication satisfy the condition

$$\lambda(a \cdot b) = (\lambda a) \cdot b = a \cdot (\lambda b), \quad \forall \lambda \in \mathbb{C}, \quad a, b \in A.$$

The multiplicatively invertible elements of an algebra are called *units*. It should be noted that $\varphi(1) = 1$ for every *algebra-homomorphism* $\varphi: A \to B$, by definition.

Thus, even in the general n-dimensional case, $\mathcal{O}(X)$ is an algebra that contains the coordinate functions z_j; with those, it must also include the polynomial algebra $\mathbb{C}[z_1, \dots, z_n]$. In fact, $\mathcal{O}(X)$ contains every function that can be constructed from some polynomial by replacing each z_j with a holomorphic function f_j of one variable.

1.2 Proposition. *$\mathcal{O}(X)$ is an algebra whose set of units $\mathcal{O}^*(X)$ consists precisely of those holomorphic functions on X with no zeros.* ■

According to a nontrivial theorem of Hartogs, a partially holomorphic function is necessarily continuous. Since we do not need that theorem, we refer the reader to the literature for its proof (see [Ha], [Hö 2.2.8]). It is easy to prove the following weaker statement:

1.3 Proposition. *A function $f: X \to \mathbb{C}$ is holomorphic iff it is locally bounded and partially holomorphic.*

(A function f is called *locally bounded* if each point of its domain has a neighborhood on which f is bounded.)

Proof of 1.3. Since we are concerned with a local statement, we may assume that f is partially holomorphic on $X = \{z \in \mathbb{C}^n; |z_j| < r, \forall j\}$ and that $M := \sup\{|f(z)|; z \in X\}$ is finite; we wish to show that f is continuous at the point $z = 0$. Observe that

$$f(z_1, \dots, z_n) - f(0, \dots, 0) = \sum_{j=1}^{n} (f(0, \dots, 0, z_j, \dots, z_n) - f(0, \dots, 0, z_{j+1}, \dots, z_n)).$$

For fixed j and fixed $z_{j+1}, \dots, z_n$, the function

$$f_j: \mathbb{C} \mapsto \mathbb{C}, \; z_j \mapsto f(0, \dots, 0, z_j, \dots, z_n) - f(0, \dots, 0, z_{j+1}, \dots, z_n),$$

is holomorphic in the variable z_j, so it follows easily from the one-dimensional version of Schwarz's Lemma (see [Co VI.2.1] or E.4h) that

$$|f_j(z_j)| \leqq 2 \frac{M}{r} |z_j|.$$

That establishes the inequality

$$|f(z_1, \dots, z_n) - f(0, \dots, 0)| \leqq 2 \frac{M}{r} \sum_{j=1}^{n} |z_j|,$$

which implies that f is continuous at $z = 0$. The reverse implication is trivial. ■

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[2] Specifically, it is a $\mathbb{C}$-algebra; as with functions, vector spaces, differentiability, manifolds, etc., we mention the field of scalars only if it is not $\mathbb{C}$!