COMPLEX VARIABLES AND APPLICATIONS

$w = f(z)$

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and
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Fourth Edition

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PREFACE

This book is a revision of the third edition, published in 1974. That edition has served, just as the first two editions did, as a textbook for a one-term introductory course in the theory and applications of functions of a complex variable. This revision preserves the basic content and style of the earlier editions, the first two of which were written by Ruel V. Churchill alone.

In this edition the authors have improved the exposition by making the examples more prominent in the text and by redrawing and adding a number of figures. The material on integrals and residues and their applications is now reached earlier, and the chapter on mapping by elementary functions leads more directly into the chapters on conformal mapping and its applications. To mention some other improvements, the sections on finding roots of complex numbers and calculating residues of functions at isolated singular points have been completely rewritten with special attention paid to the understanding of concepts and less reliance on formulas.

As was the case with the earlier editions, the first objective of this edition is to develop in a rigorous and self-contained manner those parts of the theory which are prominent in the applications of the subject. The second objective is to furnish an introduction to applications of residues and conformal mapping. Special emphasis is given to the use of conformal mapping in solving boundary value problems which arise in studies of heat conduction, electrostatic potential, and fluid flow. Hence the book may be considered as a companion volume to the authors' "Fourier Series and Boundary Value Problems" and Ruel V. Churchill's "Operational Mathematics," in which other classical methods for solving boundary value problems are treated. The latter book also contains applications of residues in connection with Laplace transforms.

The first nine chapters of this book, with various substitutions from the remaining chapters, have for many years formed the content of a 3-hour course given each term at The University of Michigan. The classes have consisted mainly of seniors and graduate students majoring in mathematics, engineering, or one of the physical sciences. The students have usually completed one term of advanced calculus. Some of the material is not covered in the lectures and is left for students to read on their own. If mapping by elementary functions and applications of conformal mapping are desired earlier in the course, one can skip to Chapters 7, 8, and 9 immediately after Chapter 3 on elementary functions.

Most of the basic results are stated as theorems, followed by examples and exercises which illustrate those results. A bibliography of other and, in many cases, more advanced books is provided in Appendix 1. A table of conformal transformations useful in applications appears in Appendix 2.

In preparing this revision, the authors have taken advantage of suggestions from a variety of people, especially Douglas G. Dickson, who suggested improvements in the treatment of antiderivatives. The authors are also indebted to their editors, John J. Corrigan, who provided a number of anonymous reviews of both the last edition and the present one in manuscript form, and Peter R. Devine, who saw this edition through its final stages.

Ruel V. Churchill
James Ward Brown

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CHAPTER

ONE

COMPLEX NUMBERS

In this chapter we survey the algebraic and geometric structure of the complex number system. We assume various corresponding properties of real numbers to be known.

1. Definition

Complex numbers z can be defined as ordered pairs

$$z = (x, y) \tag{1}$$

of real numbers x and y, with operations of addition and multiplication to be specified below. It is customary to identify the pairs $(x, 0)$ with the real numbers x. The set of complex numbers thus includes the real numbers as a subset. Complex numbers of the form $(0, y)$ are called *pure imaginary numbers*. The real numbers x and y in expression (1) are known as the *real and imaginary parts* of z, respectively; and we write

$$\operatorname{Re} z = x, \qquad \operatorname{Im} z = y. \tag{2}$$

Two complex numbers (x_1, y_1) and (x_2, y_2) are said to be *equal* whenever they have the same real parts and the same imaginary parts. That is,

$$(x_1, y_1) = (x_2, y_2) \qquad \text{if and only if} \qquad x_1 = x_2 \text{ and } y_1 = y_2. \tag{3}$$

The *sum* $z_1 + z_2$ and *product* z_1z_2 of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are defined by the equations

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \tag{4}$$

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, y_1x_2 + x_1y_2). \tag{5}$$

In particular, $(x, 0) + (0, y) = (x, y)$ and $(0, 1)(y, 0) = (0, y)$; hence

$$(x, y) = (x, 0) + (0, 1)(y, 0). \tag{6}$$

Note that the operations defined by equations (4) and (5) become the usual operations of addition and multiplication when restricted to the real numbers:

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0),$$

$$(x_1, 0)(x_2, 0) = (x_1 x_2, 0).$$

The complex number system is thus a natural extension of the real number system.

Thinking of a real number as either x or $(x, 0)$ and *letting i denote the pure imaginary number* $(0, 1)$, we can rewrite equation (6) as

$$(x, y) = x + iy.^* \tag{7}$$

Also, with the convention $z^2 = zz$, $z^3 = z^2 z$, etc., we find that

$$i^2 = (0, 1)(0, 1) = (-1, 0);$$

that is,

$$i^2 = -1.$$

In view of expression (7), equations (4) and (5) become

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2), \tag{8}$$

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2). \tag{9}$$

Observe that the right-hand sides of these equations can be obtained by formally manipulating the terms on the left as if they involved only real numbers and by replacing i^2 by -1 when it occurs.

2. Algebraic Properties

Various properties of addition and multiplication of complex numbers are the same as for real numbers. We list here the more basic of these algebraic properties and verify a few of them.

The commutative laws

$$z_1 + z_2 = z_2 + z_1, \qquad z_1 z_2 = z_2 z_1 \tag{1}$$

and the associative laws

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \qquad (z_1 z_2) z_3 = z_1 (z_2 z_3) \tag{2}$$

follow easily from the definitions of addition and multiplication of complex numbers and the fact that real numbers obey these laws. For example, if

$$z_1 = (x_1, y_1) \qquad \text{and} \qquad z_2 = (x_2, y_2),$$

then

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1)$$
$$= (x_2, y_2) + (x_1, y_1) = z_2 + z_1.$$

Verification of the rest of the above laws, as well as the distributive law

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3, \tag{3}$$

is left to the exercises.

*In electrical engineering the symbol j is used instead of i.

According to the commutative law for multiplication, $iy = yi$; hence it is permissible to write either

$$z = x + iy \qquad \text{or} \qquad z = x + yi.$$

Also, because of the associative laws, a sum $z_1 + z_2 + z_3$ or a product $z_1 z_2 z_3$ is well defined without parentheses, just as it is with real numbers.

The additive identity $0 = (0, 0)$ and the multiplicative identity $1 = (1, 0)$ for real numbers carry over to the entire complex number system. That is,

$$(4) \qquad z + 0 = z \qquad \text{and} \qquad z \cdot 1 = z$$

for every complex number z. Furthermore, 0 and 1 are the only complex numbers with such properties. To establish the uniqueness of 0, we suppose that (u, v) is an additive identity, and we write

$$(x, y) + (u, v) = (x, y)$$

where (x, y) is any complex number. It follows that

$$x + u = x \qquad \text{and} \qquad y + v = y;$$

that is, $u = 0$ and $v = 0$. The complex number $0 = (0, 0)$ is therefore the only additive identity. A similar method can be used to show that 1 is unique as a multiplicative identity.

There is associated with each complex number $z = (x, y)$ an additive inverse

$$(5) \qquad -z = (-x, -y)$$

which satisfies the equation $z + (-z) = 0$. Moreover, there is only one additive inverse for any given z. Additive inverses are used to define subtraction:

$$(6) \qquad z_1 - z_2 = z_1 + (-z_2).$$

So if $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then

$$(7) \qquad z_1 - z_2 = (x_1 - x_2, y_1 - y_2) = (x_1 - x_2) + i(y_1 - y_2).$$

Likewise, for any *nonzero* complex number $z = (x, y)$, there is a number z^{-1} such that $zz^{-1} = 1$. This multiplicative inverse is less obvious than the additive one. To find it, we seek real numbers u and v, expressed in terms of x and y, such that

$$(x, y)(u, v) = (1, 0).$$

According to equation (5), Sec. 1, which defines the product of two complex numbers, u and v must satisfy the pair

$$xu - yv = 1, \qquad yu + xv = 0$$

of linear simultaneous equations; and simple computation yields the unique solution

$$u = \frac{x}{x^2 + y^2}, \qquad v = \frac{-y}{x^2 + y^2}.$$

The multiplicative inverse of $z = (x, y)$ is, then,

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right) \qquad (z \neq 0). \tag{8}$$

The existence of multiplicative inverses enables us to show that *if a product* z_1z_2 *is zero, then so is at least one of the factors* z_1 *and* z_2. For suppose that $z_1z_2 = 0$ and $z_1 \neq 0$. The inverse z_1^{-1} exists; and, according to the definition of multiplication, any complex number times zero is zero. Hence

$$z_2 = 1 \cdot z_2 = (z_1^{-1}z_1)z_2 = z_1^{-1}(z_1z_2) = z_1^{-1} \cdot 0 = 0. \tag{9}$$

That is, if $z_1z_2 = 0$, either $z_1 = 0$ or $z_2 = 0$; or possibly both z_1 and z_2 equal zero. Another way to state this result is that *if two complex numbers* z_1 *and* z_2 *are nonzero, then so is their product* z_1z_2.

Division by a nonzero complex number is defined:

$$\frac{z_1}{z_2} = z_1z_2^{-1} \qquad (z_2 \neq 0). \tag{10}$$

If $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, equations (8) and (10) show that

$$\frac{z_1}{z_2} = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}, \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right) = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right) \qquad (z_2 \neq 0). \tag{11}$$

The quotient z_1/z_2 is not defined when $z_2 = 0$; note that $z_2 = 0$ means that $x_2^2 + y_2^2 = 0$, and this is not permitted in expressions (11).

Finally, we mention some useful identities involving quotients. They are based on the relation

$$\frac{1}{z_2} = z_2^{-1} \qquad (z_2 \neq 0), \tag{12}$$

which is equation (10) when $z_1 = 1$ and which allows us to write that equation in the form

$$\frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right) \qquad (z_2 \neq 0). \tag{13}$$

Noticing that (see Exercise 11)

$$(z_1z_2)(z_1^{-1}z_2^{-1}) = (z_1z_1^{-1})(z_2z_2^{-1}) = 1 \qquad (z_1 \neq 0,\ z_2 \neq 0),$$

and hence that $(z_1z_2)^{-1} = z_1^{-1}z_2^{-1}$, one can use relation (12) to verify the identity

$$\frac{1}{z_1z_2} = \left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right) \qquad (z_1 \neq 0,\ z_2 \neq 0). \tag{14}$$

With the aid of equations (13) and (14), it is then easy to show that

$$\frac{z_1 + z_2}{z_3} = \frac{z_1}{z_3} + \frac{z_2}{z_3}, \qquad \frac{z_1z_2}{z_3z_4} = \left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) \qquad (z_3 \neq 0,\ z_4 \neq 0). \tag{15}$$

Example. Computations such as the following are now justified:

$$\left(\frac{1}{2-3i}\right)\left(\frac{1}{1+i}\right)=\frac{1}{5-i}=\left(\frac{1}{5-i}\right)\left(\frac{5+i}{5+i}\right)=\frac{5+i}{26}=\frac{5}{26}+\frac{1}{26}i.$$

EXERCISES

1. Verify that

   (*a*) $(\sqrt{2}-i)-i(1-\sqrt{2}i)=-2i$; (*b*) $(2,-3)(-2,1)=(-1,8)$;

   (*c*) $(3,1)(3,-1)\left(\frac{1}{5},\frac{1}{10}\right)=(2,1)$; (*d*) $\dfrac{1+2i}{3-4i}+\dfrac{2-i}{5i}=-\dfrac{2}{5}$;

   (e) $\dfrac{5}{(1-i)(2-i)(3-i)}=\dfrac{1}{2}i$; (*f*) $(1-i)^4=-4$.

2. Verify that each of the two numbers $z=1\pm i$ satisfies the equation $z^2-2z+2=0$.
3. Solve the equation $z^2+z+1=0$ for $z=(x,y)$ by writing

   $$(x,y)(x,y)+(x,y)+(1,0)=(0,0)$$

   and then solving a pair of simultaneous equations in x and y.

   Suggestion: Note that $y\neq 0$ since no real number x satisfies the equation

   $$x^2+x+1=0.$$

   Ans. $z=\left(-\dfrac{1}{2},\pm\dfrac{\sqrt{3}}{2}\right)$.

4. Prove that multiplication is commutative, as stated in the second of equations (1), Sec. 2.
5. Verify the associative laws (2), Sec. 2.
6. Verify the distributive law (3), Sec. 2.
7. Apply laws established in Exercises 5 and 6 to show that

   $$z(z_1+z_2+z_3)=zz_1+zz_2+zz_3.$$

8. Show that the complex number $1=(1,0)$ is the only multiplicative identity.
9. Show that $-z=(-x,-y)$ is the only additive inverse of a given complex number $z=(x,y)$.
10. Prove that

    (*a*) $\text{Im}(iz)=\text{Re}\,z$; (*b*) $\text{Re}(iz)=-\text{Im}\,z$; (*c*) $1/(1/z)=z\quad(z\neq 0)$;

    (*d*) $(-1)z=-z$.

11. Use the associative and commutative laws for multiplication to show that

    $$(z_1z_2)(z_3z_4)=(z_1z_3)(z_2z_4).$$

12. Prove that if $z_1z_2z_3=0$, then at least one of the three factors is zero.
13. Verify identity (14), Sec. 2.
14. Establish the first of identities (15), Sec. 2.
15. Establish the second of identities (15), Sec. 2, and use it to prove the cancellation law

$$\frac{zz_1}{zz_2} = \frac{z_1}{z_2} \qquad (z \neq 0,\ z_2 \neq 0).$$

16. Show that $(1 + z)^2 = 1 + 2z + z^2$.

17. Use mathematical induction to establish the binomial formula

$$(z_1 + z_2)^n = z_1^n + \frac{n}{1!}z_1^{n-1}z_2 + \frac{n(n-1)}{2!}z_1^{n-2}z_2^2 + \cdots$$
$$+ \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}z_1^{n-k}z_2^k + \cdots + z_2^n,$$

where z_1 and z_2 are any two complex numbers and n is a positive integer $(n = 1, 2, \ldots)$.

3. Geometric Interpretation

It is natural to associate the complex number $z = x + iy$ with a point in the plane whose cartesian coordinates are x and y. Each complex number corresponds to just one point, and conversely. The number $-2 + i$, for instance, is represented by the point $(-2, 1)$ in Fig. 1. The number z can also be thought of as the directed line

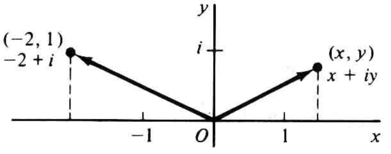

Figure 1

segment, or vector, from the origin to the point (x, y). In fact, we often refer to a complex number z as the point z or the vector z. When used for the purpose of displaying the numbers $z = x + iy$ geometrically, the xy plane is called the *complex plane*, or the *z plane*. The x axis is called the *real axis*, and the y axis is known as the *imaginary axis*.

According to the definition of the sum of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the number $z_1 + z_2$ corresponds to the point $(x_1 + x_2, y_1 + y_2)$. It also corresponds to a vector with those coordinates as its components. Hence $z_1 + z_2$ may be obtained vectorially as shown in Fig. 2. The difference $z_1 - z_2 = z_1 + (-z_2)$ corresponds to the sum of the vectors for z_1 and $-z_2$ (Fig. 3). Note that the number $z_1 - z_2$ can also be interpreted as the directed line segment from the point (x_2, y_2) to the point (x_1, y_1).

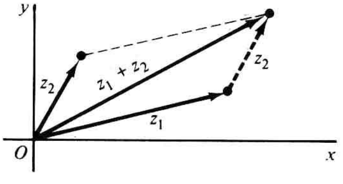

Figure 2

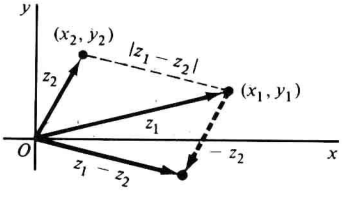

Figure 3

Although the product of two complex numbers z_1 and z_2 is itself a complex number represented by a vector, that vector lies in the same plane as the vectors for z_1 and z_2. Evidently, then, this product is neither the scalar nor the vector product used in ordinary vector analysis. The geometric interpretation of the product of z_1 and z_2 is discussed in Sec. 5.

The *modulus,* or *absolute value,* of a complex number $z = x + iy$ is defined as the nonnegative real number $\sqrt{x^2 + y^2}$ and is denoted by $|z|$; that is,

$$|z| = \sqrt{x^2 + y^2}. \tag{1}$$

Geometrically, the number $|z|$ is the distance between the point (x, y) and the origin, or the length of the vector representing z. It reduces to the usual absolute value in the real number system when $y = 0$. Note that while *the inequality $z_1 < z_2$ is meaningless unless both z_1 and z_2 are real,* the statement $|z_1| < |z_2|$ means that the point z_1 is closer to the origin than the point z_2 is.

Example 1. Since $|-3 + 2i| = \sqrt{13}$ and $|1 + 4i| = \sqrt{17}$, the point $-3 + 2i$ is closer to the origin than $1 + 4i$ is.

The distance between two points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is $|z_1 - z_2|$. This is clear from Fig. 3, since $|z_1 - z_2|$ is the length of the vector representing $z_1 - z_2$. Alternatively, it follows from definition (1) and the expression

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

that

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

The complex numbers z corresponding to the points lying on the circle with center z_0 and radius R thus satisfy the equation $|z - z_0| = R$, and conversely. We refer to this set of points simply as the circle $|z - z_0| = R$.

Example 2. The equation $|z - 1 + 3i| = 2$ represents the circle whose center is $z_0 = (1, -3)$ and whose radius is $R = 2$.

It also follows from definition (1) that the real numbers $|z|$, $\text{Re } z = x$, and $\text{Im } z = y$ are related by the equation

$$|z|^2 = (\text{Re } z)^2 + (\text{Im } z)^2, \tag{2}$$

as well as the inequalities

$$|z| \geqq |\text{Re } z| \geqq \text{Re } z, \qquad |z| \geqq |\text{Im } z| \geqq \text{Im } z. \tag{3}$$

The *complex conjugate,* or simply the conjugate, of a complex number $z = x + iy$ is defined as the complex number $x - iy$ and is denoted by $\bar{z}$; that is,

$$\bar{z} = x - iy. \tag{4}$$

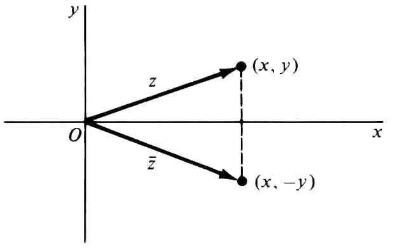

Figure 4

The number $\bar{z}$ is represented by the point $(x, -y)$, which is the reflection in the real axis of the point (x, y) representing z (Fig. 4). Note that $\bar{\bar{z}} = z$ and $|\bar{z}| = |z|$ for all z.

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\overline{z_1 + z_2} = (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2).$$

So the conjugate of the sum is the sum of the conjugates:

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2. \tag{5}$$

In like manner, it is easy to show that

$$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2, \tag{6}$$

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \tag{7}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \qquad (z_2 \neq 0). \tag{8}$$

The sum $z + \bar{z}$ of a complex number $z = x + iy$ and its conjugate $\bar{z} = x - iy$ is the real number $2x$, and the difference $z - \bar{z}$ is the pure imaginary number $2iy$. Hence we have the identities

$$\operatorname{Re} z = \frac{z + \bar{z}}{2}, \qquad \operatorname{Im} z = \frac{z - \bar{z}}{2i}. \tag{9}$$

An important identity relating the conjugate of a complex number $z = x + iy$ to its modulus is

$$z\bar{z} = |z|^2, \tag{10}$$

where each side is equal to $x^2 + y^2$. It provides another way of determining the quotient z_1/z_2 in expressions (11), Sec. 2. The procedure is to multiply both numerator and denominator by $\bar{z}_2$ so that the denominator becomes the real number $|z_2|^2$.

Example 3. As an illustration,

$$\frac{-1 + 3i}{2 - i} = \frac{(-1 + 3i)(2 + i)}{(2 - i)(2 + i)} = \frac{-5 + 5i}{|2 - i|^2} = \frac{-5 + 5i}{5} = -1 + i.$$

Also, see the example at the end of Sec. 2.