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LOGIC FOR MATHEMATICIANS

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PREFACE

The present text is just what its title claims, namely, a text on logic written for the mathematician. The text starts from first principles, not presupposing any previous specific knowledge of formal logic, and tries to cover thoroughly all logical questions which are of interest to a practicing mathematician.

We use symbolic logic in the present text, because we do not know how otherwise to attain the desired precision. Any reader of the text must perforce become a competent operator in symbolic logic. However, for us this is only a means to an end, and not an end in itself. Indeed, to mitigate the difficulties of learning and operating the symbolic logic, we have introduced some novelties. These may be of interest to students of logic, but introduction of logical novelties is not any part of the aim of this text. We seek to convey to mathematicians a precise knowledge of the logical principles which they use in their daily mathematics, and to do so as quickly as possible. In this respect, we feel that the present text is unique.

Modern logic has become a large and diversified field of study, with many well-developed branches. Many of these branches have little value to the mathematician as a tool for mathematical reasoning. A text on such a branch of logic, however excellent, would be of little interest to a reader who is primarily a mathematician. Contrariwise, certain topics of great value as tools for mathematical reasoning have little interest for students of logic, and are almost never treated in books on logic. Thus it happens that among the many books on logic, none is completely suitable for the mathematician.

One of the most suitable is the epoch-making "Principia Mathematica" of Whitehead and Russell. The subject matter in "Principia Mathematica" was admirably chosen for the needs of mathematicians, and we have followed this text closely with regard to subject matter. We have omitted a few topics which seem to be little used nowadays, and instead have included treatments of such new developments as Zorn's lemma. We have improved on the symbolic machinery of "Principia Mathematica," which is out of date and extremely unwieldy. By using techniques invented since its writing, we have succeeded in condensing most of "Principia Mathematica's" three large volumes into the present text.

Since familiar logical principles often look very strange in the garb of symbolic logic, we have included a large number of pertinent examples of ordinary mathematical reasoning handled by symbolic means. This should help the reader to apply the principles of this text to his own problems in mathematical reasoning.

Although the present text is complete and does not presuppose any previous acquaintance with logic, it is written for the mathematician with some maturity. For one thing, the illustrative examples are chosen from a variety of fields of mathematics, and their point will be lost on the mathematically immature.

By including numerous exercises, we have tried to make this text suitable for classroom instruction, and have used it this way ourselves. With a teacher to help, less maturity is needed on the part of the reader than if he is reading it alone. However, even with a teacher to help, it is recommended that the student should have had some mathematics beyond the calculus, preferably a course in which some attention was paid to careful mathematical reasoning. Let us recall that this text attempts to treat all logical principles which are useful in modern mathematics, and unless the reader has some acquaintance with the mathematical fields in which the principles are to be used, he will find a study of the principles alone rather sterile. It is in the hope of counteracting such sterility that we have included so many illustrations of reasoning from standard mathematical texts.

We are vastly indebted to the many logicians with whom we have been associated in the past twenty years as well as to the many others whose writings we have read. This debt is only partially indicated by the titles in our bibliography. Almost equally important for the present text have been the many suggestions from mathematicians who are not primarily logicians, but who have been kind enough to tell us of logical questions which they would like to see answered. We hope that they will find them answered in the present text.

Those theorems, or parts of theorems, or corollaries, which are referred to at least five times in later sections are marked with a *. Those of particular importance are marked with a **.

At the end of the present text there is a bibliography arranged alphabetically according to the names of the authors. References to items having a single author are made by giving the author's name and the date of the item, as "Hardy, 1947." In the one case where this is ambiguous, we use "Zermelo, 1908, first paper" and "Zermelo, 1908, second paper." References to items having two authors are made by giving the names of the authors, as "Hardy and Wright."

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CHAPTER I

WHAT IS SYMBOLIC LOGIC?

1. A Hypothetical Interview. We wish to record an imaginary interview between a modern mathematician and one of past times. Our mathematician of the past will be Descartes, but we should like to leave our modern mathematician anonymous; in the classic tradition of mathematics, we shall refer to him as Professor X. We imagine Professor X equipped with a time-traveling machine, so that he can go back to chosen points in time and interview various famous mathematicians of the past. Professor X elects to go back to a time just after the invention of coordinate geometry by Descartes and to have an interview with Descartes about his new invention. Professor X takes with him a gift of several reams of coordinate paper, together with a supply of mechanical pencils and erasers, which so impress Descartes that he is very cordial. They discourse on many matters, of which we shall record only their discussion of continuous curves.

They define a curve as continuous if it can be drawn without lifting the pencil from the paper. Descartes, fascinated with his pencils and paper, draws a large number of curves and classifies them into continuous and discontinuous. Fortified with his knowledge of early twentieth century mathematics, Professor X is able to suggest many interesting curves and even manages to trick Descartes at first with some special curves like

$$y = \frac{\sin x}{x},$$

which is not defined at $x = 0$ and so has a gap there which makes it discontinuous. However, Descartes, being a clever mathematician, soon catches on to all Professor X's tricks and can quickly and unerringly classify even the most complicated curves as continuous or discontinuous.

Needless to say, Professor X is familiar with the modern precise definition of continuity:

(1) A function f is continuous at $x = a$ if $f(a)$ is defined and unique, and if for each positive ε there is a positive δ such that whenever $|x - a| < \delta$ it follows that $|f(x) - f(a)| < \varepsilon$.

(2) A function f is continuous if it is continuous at $x = a$ for each value of a.

Professor X decides to acquaint Descartes with this definition with the intention of persuading him to adopt it in place of the vague intuitive idea of tracing a curve without lifting the pencil from the paper. He decides further that he cannot argue in favor of his precise ε-δ definition on the basis that it is more useful for deciding whether a curve (or function) is continuous. Already, with his vague definition of continuity, Descartes can decide which curves are continuous and can do so quickly and correctly. As a matter of fact, Professor X realizes that he himself usually decides whether a function is continuous by visualizing if its graph can be drawn with a continuous pencil stroke and only uses the ε-δ definition of continuity to prove the conclusion which he has reached by visualizing the graph. Clearly then, the value of the ε-δ definition lies mainly in proving things about continuity and only slightly in deciding things about continuity. Professor X reflects that the situation is quite analogous to that in early twentieth century mathematical circles where, if one has a difficult mathematical problem, one is apt to proceed quite intuitively, interchanging limits of integration, differentiating under the integral sign, etc., in hopes of guessing an answer. Only after one has guessed an answer, and wishes to verify it beyond doubt, does one bring in the precise definitions, the ε's and δ's, and the other powerful machinery of modern mathematics. For getting answers, it is better to use intuitive arguments, even rather vague ones. For proving answers, only rigid, formal arguments can be trusted.

Professor X thinks of an analogous situation which he can present to Descartes. For the Egyptian originators of geometry, geometric concepts were quite vague. A straight line was a stretched string; parallel lines were wagon tracks; etc. This vagueness did not prevent the Egyptians from discovering many useful geometric theorems but made it quite impossible for them to prove them. However, the Greeks introduced the precise ideas of abstract straight lines, etc., and were thus enabled to devise proofs of geometric theorems. The great increase of geometric knowledge with the Greeks makes it hard to believe that the increased precision was not also of value in discovering geometric theorems as well as proving them.

Actually, Professor X found Descartes very agreeable to his suggestions and quite willing to replace his vague idea of continuity by a precise one. However, Descartes raised one difficulty which Professor X had not foreseen. Descartes put it as follows.

"I have here an important concept which I call continuity. At present my notion of it is rather vague, not sufficiently vague that I cannot decide which curves are continuous, but too vague to permit of careful proofs. You are proposing a precise definition of this same notion. However, since my definition is too vague to be the basis for a careful proof, how are we going to verify that my vague definition and your precise definition are definitions of the same thing?"

If by "verify" Descartes meant "prove," it obviously could not be done, since his definition was too vague for proof. If by "verify" Descartes meant "decide," then it might be done, since his definition was not too vague for purposes of coming to decisions. Actually, Descartes and Professor X did finally decide that the two definitions were equivalent, and they arrived at the decision as follows. Descartes had drawn a large number of curves and classified them into continuous and discontinuous, using his vague definition of continuity. He and Professor X checked through all these curves and classified them into continuous and discontinuous using the ϵ-δ definition of continuity. Both definitions gave the same classification. As these were all the interesting curves that either of them had been able to think of, the evidence seemed "conclusive" that the two definitions were equivalent.

2. The Role of Symbolic Logic. When Professor X returned to the present, he related these matters to us. We said that we were reminded of the situation with respect to symbolic logic. Professor X suggested that, as he knew nothing about symbolic logic, the connection could hardly be apparent to him, and he asked if we could explain without getting too complicated. We replied as follows.

Suppose Professor X wishes to prove that from assumption A he can deduce conclusion Z. How does he proceed? The most straightforward way is to observe that B is a logical consequence of A, then C is a logical consequence of B, and so on until he comes to Z. For this, it is required not only that Professor X be able to discover the sequence of statements $B, C, \ldots,$ but that he be able to decide that each is a logical consequence of the preceding. One thing that symbolic logic does is give a precise definition of when one statement is a logical consequence of another statement. To get the connection with Descartes, we set up an analogy as follows. A step of Professor X's proof (such as deducing B from A, or C from B, etc.) is to correspond to one of Descartes's curves. Deciding whether the step is logically correct or not is to correspond to deciding whether the curve is continuous or not. To continue the analogy, we note that Professor X is quite skillful at deciding when a step is logically correct, just as Descartes was quite skillful at deciding when a curve is continuous. Moreover, Professor X bases his decisions on a rather vague intuitive notion of logical correctness, just as Descartes based his decisions on a rather vague intuitive notion of continuity. Furthermore, the vague intuitive notion of logical correctness is adequate for deciding about the correctness of a logical step, just as the vague intuitive notion of continuity was adequate for deciding about the continuity of a curve. If one wishes to *prove* the correctness of a logical step, a precise definition of logical correctness will be needed, just as a precise definition of continuity was needed before

Descartes could prove a curve to be continuous. Finally, symbolic logic furnishes a precise definition of logical correctness and so is analogous to the ε-δ definition of continuity, which furnishes a precise definition of continuity.

"Why do you think my notion of logical correctness is rather vague and intuitive?" asked Professor X. "I admit that I very seldom justify the logic involved in my proofs, but that doesn't prove that I can't. After all, I took two years of good stiff courses in logic under the chairman of the philosophy department back in '27–'29."

Our reply was that classical logic was quite inadequate for mathematical reasoning, being particularly weak in treating functions, use of infinite classes, and other matters of great importance in mathematics. As a matter of fact, the first treatment of logic adequate for use in modern mathematics was the famous "Principia Mathematica" of Whitehead and Russell (see Whitehead and Russell).

Professor X admitted that his two years of logic had been of very little use in mathematics. He further admitted that he had no notion how to give a precise definition of logical correctness. Nevertheless, he had always been able to tell which proofs were valid and which were not. What would he gain by learning a precise definition of logical correctness?

We countered by referring him back to his interview with Descartes. What would he have said if Descartes had answered in similar fashion that he had been getting along very well with a vague definition of continuity and had no need of a precise definition?

This seemed to satisfy Professor X. However, he had one further question to ask.

"I should like to ask the same question that Descartes asked. You are proposing to give a precise definition of logical correctness which is to be the same as my vague intuitive feeling for logical correctness. How do you intend to show that they are the same?"

This is not merely Professor X's question. It should be the question of every reader of the present text.

Actually, not all mathematicians have exactly the same notion of logical correctness. Mathematics is a living, growing subject, and mathematicians do not all work in the same branch of mathematics. Often mathematicians in one branch of mathematics make constant use of some logical principle which is regarded with distrust by mathematicians in other branches. The axiom of choice, to which we shall devote a chapter of discussion, is such a principle.

However, there is a sort of "common denominator" of notions of logical correctness, and we claim to give a symbolic logic which is a precise definition of logical correctness which agrees with this "common denominator."

Our symbolic logic is accordingly incomplete. In the case of a principle like the axiom of choice, which is in dispute among mathematicians, our symbolic logic deliberately fails to classify it as either correct or incorrect, leaving the individual reader free to make whichever decision pleases him most. However, we do attempt to convince the reader that logical principles which are judged correct by the great majority of mathematicians are classified as correct by our symbolic logic and that principles which are judged incorrect by the great majority of mathematicians are classified as incorrect by our symbolic logic.

Our procedure for doing this has already been foreshadowed in the interview between Descartes and Professor X. Just as they decided to accept the equivalence of the intuitive and precise definitions of continuity because these definitions agreed in a large number of cases, even so a reader might be convinced that our symbolic logic agrees with his intuitive notions of logical correctness if he is shown that they agree in a large number of cases. Accordingly, we shall give a large number and wide variety of illustrations of mathematical reasoning and show how to classify each as correct or incorrect on the basis of our symbolic logic. We have tried to choose our illustrations from well-known sources, so that there would be no doubt about the general opinion of mathematicians as to the correctness or incorrectness of the reasoning in our illustrations. With the general opinion on the correctness agreeing with our symbolic logic in a wide variety of cases, we feel that most readers will be convinced.

For the benefit of any professional skeptics, we admit here and now that certainly no number of illustrations could ever suffice to carry absolute conviction.

The symbolic logic which we present is a modernized version of that presented in the "Principia Mathematica" of Whitehead and Russell. We have altered the form of the system somewhat, using a greatly simplified version of the theory of types due to Quine (see Quine, 1937). Minor details have been adjusted to bring them into line with common mathematical usage. Simplifications and improvements of the proofs have been adopted from numerous sources. We have not attempted to list these sources, since in the present text we are not concerned with the genesis of the logic but with its applications. Persons interested in the connections of this symbolic logic with others may consult such works as Hilbert and Bernays; Church, 1944; Quine, 1951; and Hilbert and Ackermann.

3. General Nature of Symbolic Logic. The aim in constructing our symbolic logic is that it shall serve as a precise criterion for determining whether or not a given instance of mathematical reasoning is correct. The symbolic logic which we shall present is primarily intended to be a tool in mathematical reasoning. Of course, many of the logical principles involved have general application outside of mathematics, but there are many fields of human endeavor in which these principles are of little value. Politics, salesmanship, ethics, and many such fields have little or no use for the sort of logic used in mathematics, and for these our symbolic logic would be quite useless. In engineering and science, particularly those branches of science which make extensive use of mathematics, the symbolic logic might be of considerable value. However, it would be fairly inadequate for the logical needs of even the most mathematical sciences. For one thing, no adequate symbolic treatment of the relationship involving cause and effect has yet been devised. However, if one is satisfied to restrict attention to purely mathematical reasoning, several quite satisfactory symbolic logics are available. We present one such in the present text.

The components of mathematical reasoning are mathematical statements. So, in building a symbolic logic, we must start with a precise definition of what a mathematical statement is. Intuitively, we can say that it is merely a declarative sentence dealing exclusively with mathematical and logical matters. Needless to say, it need not be true. "3 is a prime" and "6 is a prime" are both mathematical statements, the first true, and the second false.

Because all existing languages are full of words with multiple or ambiguous meanings, it was found necessary to construct a complete new language in order to be able to give a precise definition of "mathematical statement." This language is called symbolic logic. In order to aid the reader in learning this new language, we shall introduce him to it gradually over several chapters. Our discussions will be rather general and descriptive at first, becoming more and more exact. Correspondingly our notion of a mathematical statement will at first be merely the vague notion of a declarative sentence but will gradually be sharpened. Finally in Chapter IX we shall have developed our symbolic logic sufficiently to be able to give a precise definition of a mathematical statement.

We shall drop the "mathematical" and henceforth refer to a mathematical statement merely as a "statement."

Once a precise definition of "statement" has been given (see Chapter IX), one can give a precise definition of "valid statement" and of "demonstration." A demonstration shall be a sequence of statements such that each statement is either already known to be valid or is an assumption or is derived from previous statements of the sequence in a specified fashion. The analogy with the usual form of mathematical demonstration is quite intentional. Certain statements, designated as "axioms," are taken to be valid, and then any other statement is called "valid" if it is the final statement in a demonstration that involves no assumptions, that is, that proceeds from axioms alone.

Our definitions of "axiom" and "demonstration" will be carefully and intentionally framed so that they depend only on the forms of the statements involved, and not in the least on the meanings. Thus the decision as to whether a statement is an axiom or whether a sequence of statements is a demonstration depends not on intelligence, but on clerical skill. One could build a machine which would be quite capable of making these decisions correctly. That is, one could build a machine which would check the logical correctness of any given proof of a mathematical theorem. That the check is mechanical does not mean that it requires no intelligence at all. There are many machines of a sufficient complexity that at least a low order of intelligence is required to match their performance. In the present case, the ability to perform simple arithmetical computations is enough to check axioms and demonstrations, as was shown by Gödel (see Gödel, 1931), who put the definitions into an arithmetical form. Thus, a person with simple arithmetical skills can check the proofs of the most difficult mathematical demonstrations, provided that the proofs are first expressed in symbolic logic. This is due to the fact that, in symbolic logic, demonstrations depend only on the forms of statements, and not at all on their meanings.

This does not mean that it is now any easier to discover a proof for a difficult theorem. This still requires the same high order of mathematical talent as before. However, once the proof is discovered, and stated in symbolic logic, it can be checked by a moron.

This complete lack of any reference to the meanings of statements in symbolic logic indicates that there is no need for them to have meanings. This allows us to introduce formulas whenever they are useful without reference to whether they are meaningful. In fact, there is a type of formula about whose meaning (if any) there is great disagreement. It happens to be a useful type of formula, and we use it frequently, not being the least bit inconvenienced by its possible lack of meaning (see Chapter VIII).

This lack of reference to meanings also enables us to evade quite a number of difficult philosophical questions. This situation is quite in line with current mathematical practice. Consider the positive integers, which are at the basis of most of mathematics. Mathematicians do not care in the least what the meanings of the positive integers are, or even if they have meanings. For the mathematician, it suffices to know what operations he is permitted to perform on the positive integers. Once this information is available, any information as to the meanings of the integers is wholly irrelevant for mathematical purposes. The same applies to real numbers, imaginary numbers, functions, or any other of the paraphernalia of mathematics.

The matter was well expressed by Lewis Carroll, long-time mathematical lecturer of Christ Church, who upon being asked to contribute to a philosophical symposium responded:

"And what mean all these mysteries to me
Whose life is full of indices and surds?
$x^2 + 7x + 53$
$= \frac{11}{3}$."

We shall not make any use of the familiar term "proposition." This is because the word "proposition" refers to the meanings of statements, and we intend to ignore the meanings (if any) of our statements. However, we shall here say a word about propositions and the problems connected with them just to show how useful it is not to have to consider these problems.

A proposition is the meaning of a statement, and one says that the statement expresses the proposition. One difficulty that arises immediately is that of deciding when two different statements express the same proposition. Sometimes it is easy. Thus "three is a prime" and "Drei ist eine Primzahl" certainly express the same proposition. However, what about "Three is a prime" and "Three is greater than unity and is not divisible by any positive integers except itself and unity"? Do they express the same proposition or equivalent propositions?

Any attempt to be precise and pay attention to meanings would involve us with such problems as the above, which are really quite irrelevant for mathematics. For mathematics, it is the form that must be considered, and the meaning can be dispensed with. Our symbolic logic will accord with this doctrine.

Actually, although one carefully builds the symbolic logic so that it can be used without reference to meaning, this does not mean that we can ignore meaning in devising our logic. We recall that our symbolic logic is intended to give a precise definition of an intuitive notion of logical correctness. So the mechanical operations of our symbolic logic, though devoid of meaning, must nevertheless manage to parallel closely the intuitive thought processes based on meaning. Clearly, then, careful attention is paid to meaning and intuitive thought processes in inventing the symbolic logic.

Now that the symbolic logic has been invented, we could present it to the reader merely as a mechanical system, without reference to the motivation which underlies it. Certainly it is intended to be used in this way. Nonetheless, the reader will find it easier to learn, remember, and use the symbolic logic if we explain to him the underlying thought processes. Consequently much of our discussion in the earlier chapters will be quite intuitive in character and not particularly precise. Gradually, as our symbolic logic crystallizes out of the intuitive background, we shall become more precise, though we shall never lose sight of our intuitive background completely even after we have finally completely defined our symbolic logic and are proceeding quite mechanically.

4. Advantages and Disadvantages of a Symbolic Logic. We have already mentioned some advantages of a symbolic logic over a simple intuitive notion of logical correctness, namely, its greater precision and its lack of reference to meanings; because of the lack of reference to meanings, many difficult philosophical problems can be evaded and mechanical checks of proofs are possible.

A symbolic logic is a formal system and as such has the advantage of objectivity which is inherent in any formal system. This can be illustrated by a reference to the origins of geometry. To the Egyptians, a straight line was a stretched string. Now two stretched strings are much alike, but not completely so, and thus one person's idea of a straight line would not coincide exactly with another person's idea. As an extreme instance, one man may be dealing with a fine silk cord, and the second man with a towrope. In this case, their "straight lines" would be quite appreciably different. Then came the Greeks, who replaced the stretched string by an abstract idea of a straight line which was defined by purely formal axioms. From that time on, the straight line has meant the same thing to all who accepted the Greeks' definition. Analogously, by means of symbolic logic we replace a person's intuitive ideas, subjectively conceived and full of personal psychological overtones, by abstract formal ideas which can be the same for all persons.

A symbolic logic uses symbols and so has the advantages arising from the use of symbols, in particular, greater ease in handling complex manipulations. This is so familiar to mathematicians that an instance is probably unnecessary. We cite one anyhow for completeness. Consider the simple problem: "Mary is now three times as old as Jane. In ten years Mary will be twice as old as Jane. How old are Mary and Jane now?" Algebraically this is almost trivial. We take the symbols M and J to stand for the present ages of Mary and Jane, getting

$$M = 3J,$$

$$M + 10 = 2(J + 10).$$

Subtraction gives $J = 10$, whence we get $M = 30$. The point is that, though this is very simple when handled by symbols, it is not particularly easy if one tries to handle it intuitively. Certainly one can get the answer by words alone, but it is so awkward to do so that variants of the above problem are actually given as simple puzzles for those not accustomed to the use of algebraic technique.

It is interesting to note that the algebraic procedure outlined above does not differ greatly from the intuitive procedure that one might use to solve the problem verbally. In other words, use of symbolic manipulations does not necessarily give one any technique for solving the problem which was not already present in the intuitive case; it merely makes the existing techniques more flexible, more effective, and more apparent. This is characteristic of the use of symbols.

When one gives a precise definition of a concept, then there arises the possibility of generalizing or varying the concept by slight alterations in the definition. Thus, as long as the early Egyptians were thinking of parallel lines as wagon tracks, there was no possibility of getting non-Euclidean geometries in which parallel lines behave in quite unfamiliar fashions. However, after the Greeks had defined parallel lines as straight lines which never meet, and Euclid had defined geometry (we call it Euclidean geometry nowadays) by specifying that parallel lines should behave essentially like wagon tracks, then one could generalize to non-Euclidean geometries by specifying other behaviors for parallel lines. Similarly, by going from the simple intuitive concept of continuity to the precise ε-δ definition, one can then introduce many variations of continuity, such as absolute continuity, semicontinuity, and upper and lower semicontinuity.

An analogous situation has already arisen in connection with symbolic logic. There are now several different systems of symbolic logic available which differ in various details. We have chosen that one which seems to us most nearly in accord with the intuitive notion of logical correctness as conceived by most mathematicians.

Thus our choice of a system of symbolic logic is arbitrary. This is a disadvantage in that later study may show our choice to have been a poor one. It is also an advantage in that if we ever become dissatisfied with our choice, we can readily change it.

The main disadvantage of a system of symbolic logic is that it is a formal system divorced from intuition. Intuition arises from experience, and so may be expected to have some foundation in fact. However, a formal system is merely a model devised by human minds to represent some facts perceived intuitively. As such, it is bound to be artificial. In some cases, the artificiality is quite clear. Thus electrical engineers are taught a system for computing currents and voltages in rotating electrical machinery by representing them as complex numbers of the form $a + bi$, $i = \sqrt{-1}$ (see Glasgow, 1936). As there is nothing imaginary about the currents and voltages, this is clearly an artificial representation. Nevertheless, its advantages outweigh its obvious artificiality.

Probably the only time that the artificiality of a formal system does any harm is when the users of the system ignore or overlook the fact that it is artificial. Thus, for two thousand years it was supposed that the physical universe was actually a Euclidean three-dimensional space. This inhibited men's thinking tremendously and was a great misfortune. Nowadays, astronomical measurements have made it seem quite likely that the universe is non-Euclidean. Though this demonstrates the artificiality of Euclidean geometry, nonetheless Euclidean geometry is still extremely useful, as useful in fact as it ever was. Thus artificiality is not a serious disadvantage if one does not lose sight of the artificiality.

From the point of view of the nonmathematician, who finds it difficult to work with symbols, use of symbols is a disadvantage. We intend the present text for mathematicians, to whom the use of symbols is quite congenial, and so make no apology for the use of symbols.

We mentioned the possibility of mechanical checking of proofs as an advantage. It is not wholly an advantage. If a person has little clerical skill, he is liable to make mistakes in his mechanical checking, and so find it of little value. On the other hand, if one relies exclusively on intuition, there is danger of overlooking some detail which appears insignificant but isn't. The truth is that the average person cannot rely exclusively on either intuition or mechanical checking. For the average person, mechanical checking is a valuable adjunct to intuition, but in doing the mechanical checking he must continually refer back to his intuition to catch clerical errors.

We summarize the above points. Although we think that the average mathematician will find that a study of symbolic logic is very helpful in carrying out mathematical reasoning, we do not recommend that he should completely abandon his intuitive methods of reasoning for exclusively formal methods. Rather, he should consider the formal methods as a supplement to his intuitive methods to provide mechanical checks of critical points, and to provide the assistance of symbolic operations in complex situations, and to increase his precision and generality. He should not forget that his intuition is the final authority, so that, in case of an irreconcilable conflict between his intuition and some system of symbolic logic, he should abandon the symbolic logic. He can try other systems of symbolic logic, and perhaps find one more to his liking, but it would be difficult to change his intuition.

CHAPTER II

THE STATEMENT CALCULUS

1. Statement Functions. As indicated in the previous chapter, we shall not proceed at once to a precise definition of a statement. We have told the reader that essentially a statement is a declarative sentence (not necessarily true) which deals exclusively with mathematical and logical matters. We shall gradually make this idea precise, but in the present chapter we shall confine our attention to certain of the very simplest ways of building statements, by use of the so-called "statement functions."

We derive all the statement functions from two basic ones, "&" and "∼". Consider two statements, "*P*" and "*Q*", of symbolic logic which are translations of the English sentences "*A*" and "*B*". Then "(*P*&*Q*)" is the statement which is a translation of "*A* and *B*" and "∼*P*" is the statement which is a translation of the negation of the sentence "*A*". If "*A*" happens to be a simple sentence, the negation would most usually be formed by inserting a "not" into "*A*" at the grammatically proper place. Thus "&" is the translation of "and", and, allowing for the difference of sentence structure, "∼" is the translation of "not". Hence we usually refer to "&" and "∼" as "and" and "not", and usually read "(*P*&*Q*)" and "∼*P*" as "*P* and *Q*" and "not *P*", respectively. However, when we wish to be very careful we refer to "&" and "∼" by their correct names "ampersand" and "curl" or "twiddle".

To illustrate, let "*P*" and "*Q*" be translations of "It is raining now" and "It is not cloudy now". Then "(*P*&*Q*)" is a translation of "It is raining now and it is not cloudy now", and "∼*P*" and "∼*Q*" are translations of "It is not raining now" and "It is cloudy now". Finally "∼(*P*&*Q*)" is a translation of "Either it is not raining now or else it is cloudy now or else it is both cloudy and not raining now" or of "It is not now both raining and not cloudy" or of some such negation of "It is raining now and it is not cloudy now".

"(*P*&*Q*)" has properties analogous to the product of two numbers in arithmetic or algebra. For this reason, it is called the logical product of "*P*" and "*Q*", which are called the factors, and is often written "(*P*.*Q*)" or simply "(*PQ*)". Also, one omits the parentheses whenever possible without ambiguity, so that it may also be written "*P*&*Q*", "*P*.*Q*", or "*PQ*". To diminish the number of possible cases of ambiguity, we agree that whenever a "$\sim$" occurs it shall affect as little as possible of what follows it. This is expressed as follows.

Convention. Any given occurrence of "$\sim$" shall have as small a scope as possible.

As an illustration, consider "$\sim PQ$" (or either of the alternative forms "$\sim P.Q$" or "$\sim P\&Q$"). According to our convention, the "$\sim$" affects "P" but not "Q", and so we understand "$\sim PQ$" to mean "$(\sim P)Q$" (or "$(\sim P).Q$" or "$(\sim P)\&Q$"). Without our convention, there would be the possibility that "$\sim PQ$" might mean "$\sim(PQ)$".

The expression "$P\sim Q$" is unambiguous even without our convention, since it clearly can mean nothing but "$P\&(\sim Q)$".

By means of "&" and "$\sim$", we can translate many other English conjunctions besides "and" into symbolic logic. As before, let "P" and "Q" be statements of symbolic logic which are translations of the English sentences "A" and "B", and let us seek to find a translation for "Either A or B". First we should agree whether we interpret "Either A or B" in the exclusive sense of "Either A or B but not both" or in the inclusive sense of "Either A or B or both". According to each of the four best unabridged dictionaries, the exclusive use is the only correct use, and the inclusive use has no justification at all in correct English. Nonetheless, in mathematics the inclusive form of "or" is very commonly used, and in everyday language, it is often not clear which is intended. In legal documents one commonly finds the inclusive "or" expressed as "A and/or B".

In some languages, there are different words for the exclusive and inclusive "or". Thus in Latin, the word "aut" denotes an exclusive "or" so that "aut" means "or—but not both", whereas "vel" denotes an inclusive "or" so that "vel" means "and/or". We shall translate both the exclusive "or" and the inclusive "or" into symbolic logic.

We take first the inclusive "or". We are seeking the interpretation of "A and/or B" or the equivalent "Either A or B or both". This statement is equivalent to denying that "A" and "B" are simultaneously false, and we use this fact to carry out the translation. That "A" is false would be translated as "$\sim P$", and that "B" is false would be translated as "$\sim Q$". That both are false would be translated as "$(\sim P)\&(\sim Q)$", which we can simplify unambiguously to "$\sim P\sim Q$". The denial of this is then translated as "$\sim(\sim P\sim Q)$". So we conclude that the translation of "Either A or B or both" is "$\sim(\sim P\sim Q)$".

To translate "Either A or B but not both", it suffices to adjoin the additional statement "not both", which certainly would be translated "$\sim(PQ)$". So the translation of "Either A or B but not both" is "$(\sim(\sim P\sim Q))\&(\sim(PQ))$", which we can simplify unambiguously to "$\sim(\sim P\sim Q)\&\sim(PQ)$" or "$\sim(\sim P\sim Q)\sim(PQ)$".