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Computer Algebra in Scientific Computing

8th International Workshop, CASC 2005
Kalamata, Greece, September 2005
Proceedings

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Preface

CASC 2005 continued a tradition — started in 1998 — of international conferences on the latest advances in the application of computer algebra systems (CASs) and methods to the solution of various problems in scientific computing.

The methods of scientific computing play an important role in research and engineering applications in the natural and the engineering sciences. The significance and impact of computer algebra methods and computer algebra systems for scientific computing has increased considerably in recent times. Nowadays, such general-purpose computer algebra systems as Maple, Magma, Mathematica, MuPAD, Singular, CoCoA and others enable their users to solve the following three important tasks within a uniform framework:

(a) symbolic manipulation;
(b) numerical computation;
(c) visualization.

The ongoing development of such systems, including their integration and adaptation to modern software environments, puts them at the forefront in scientific computing and enables the practical solution of many complex applied problems in the domains of natural sciences and engineering.

Greece offers excellent infrastructures for hosting international conferences, and this was a reason for us to choose the city of Kalamata, Greece, as the location for CASC 2005, the eighth conference in the sequence of CASC conferences. The seven earlier CASC conferences, CASC 1998, CASC 1999, CASC 2000, CASC 2001, CASC 2002, CASC 2003, and CASC 2004 were held, respectively, in St. Petersburg, Russia, in Munich, Germany, in Samarkand, Uzbekistan, in Konstanz, Germany, in the Crimea (Ukraine), in Passau (Germany), and in St. Petersburg, Russia, and they proved to be successful.

The Program Committee did a tremendous job reading and evaluating 75 submitted papers, as well as soliciting external reviews, and all of this in a very short period of time. There were about three reviews per submission on average. The result of this job is reflected in this volume, which contains revised versions of the accepted papers. The collection of papers included in the proceedings covers various topics of computer algebra methods, algorithms, and software applied to scientific computing:

1. algebraic methods for nonlinear polynomial equations and inequalities;
2. symbolic-numeric methods for differential and differential-algebraic equations;
3. algorithmic and complexity considerations in computer algebra;
4. algebraic methods in geometric modelling;
5. aspects of computer algebra programming languages;
6. automatic reasoning in algebra and geometry;

7. complexity of algebraic problems;
8. exact and approximate computation;
9. parallel symbolic-numeric computation;
10. Internet accessible symbolic and numeric computation;
11. problem-solving environments;
12. symbolic and numerical computation in systems engineering and modelling;
13. computer algebra in industry;
14. solving problems in the natural sciences;
15. numerical simulation using computer algebra systems; and
16. mathematical communication.

This workshop, like the earlier CASC workshops, was intended to provide a forum for researchers and engineers in the fields of mathematics, computer science, numerical analysis, and industry, to interact and exchange ideas. An important goal of the workshop was to bring together all these specialists for the purpose of fostering progress on current questions and problems in advanced scientific computing.

CASC 2005 featured two satellite workshops

- *Algebraic and Matrix Computation with Applications*, organized by I.Z. Emiris, B. Mourrain, and M.N. Vrahatis

- *Kalamata Combinatorics*, organized by I.S. Kotsireas and C. Koukouvinos

Researchers from France, Germany, Italy, Greece, Spain, Russia, Japan, USA, Canada, Czech Republic, and Egypt participated in CASC 2005.

CASC 2005 wishes to acknowledge generous support from sponsors:

- Hellenic Ministry of Culture, *ΥΠΠΟ*, Athens, Greece
- Maplesoft, Waterloo, Ontario, Canada
- National and Kapodistrian University of Athens, Greece
- University of Patras, Greece
- Wilfrid Laurier University, Waterloo, Ontario, Canada

Our particular thanks are due to the CASC 2005 conference chairs and members of the Local Organizing Committee I.Z. Emiris (Athens), I.S. Kotsireas (Waterloo), and M.N. Vrahatis (Patras), who ably handled local arrangements in Kalamata. We also thank the members of the General Organizing Committee, W. Meixner and A. Schmidt, in particular for their work in preparing the conference proceedings.

Munich, July 2005

V.G. Ganzha
E.W. Mayr
E.V. Vorozhtsov

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CASC 2005 was organized jointly by the Department of Informatics at the Technische Universität München, Germany, and the Computer Algebra Research Group of Wilfried Launer University, Canada.

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Table of Contents

On Regular and Logarithmic Solutions of Ordinary Linear Differential Systems*

S.A. Abramov[1], M. Bronstein[2], and D.E. Khmelnov[1]

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Abstract. We present an approach to construct all the regular solutions of systems of linear ordinary differential equations using the desingularization algorithm of Abramov & Bronstein (2001) as an auxiliary tool. A similar approach to find all the solutions with entries in $C(z)[\log z]$ is presented as well, together with a new hybrid method for constructing the denominator of rational and logarithmic solutions.

1 Introduction

Let C be an algebraically closed field of characteristic 0, z be an indeterminate over C, and

$$L = Q_\rho(z)D^\rho + \cdots + Q_1(z)D + Q_0(z), \tag{1}$$

where $D = d/dz$ and $Q_\rho(z), \ldots, Q_0(z) \in C[z]$. A *regular solution* of $Ly = 0$ (or of L) at a given point $z_0 \in C$, is a solution of the form $(z - z_0)^\lambda F(z)$ with $F(z) \in C((z - z_0))[\log(z - z_0)]$, where $C((z - z_0))$ is the field of (formal) Laurent series over C. If $F(z)$ has valuation 0, then λ is called the *exponent* of the regular solution (otherwise it is an exponent modulo $\mathbb{Z}$). Using the change of variable $\bar{z} = z - z_0$, we can assume without loss of generality that $z_0 = 0$. The problem of constructing all the regular solutions is solved by the Frobenius algorithm (1873, [8, Chap.IV],[9],[14, Chap.V]), which is based on the indicial equation $f(\lambda) = 0$ of L at 0. Not only the roots of $f(\lambda) = 0$, each taken separately, are substantial for the Frobenius algorithm, but also their multiplicities and whether some roots differ by integers. Later, in 1894, L. Heffter proposed another algorithm to solve the same problem ([10, Kap.II,VIII],[14, Chap.V]). For a given root λ of the indicial equation, Heffter's algorithm constructs a basis (possibly empty) for all the regular solutions with exponent λ. Once λ is fixed, that algorithm does not depend on the multiplicity of λ, nor on the existence of another root at an integer distance from λ. It constructs a sequence $E_0, E_1, \ldots$ of linear differential equations, whose right-hand side contains solutions of the preceding equations. If

$$z^\lambda \left(g_0(z) + g_1(z)\frac{\log z}{1!} + g_2(z)\frac{\log^2 z}{2!} + \cdots + g_m(z)\frac{\log^m z}{m!} \right)$$

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is a regular solution of (1) then $g_i(z) \in C((z))$ is a solution of E_i for each i. All the regular solutions of (1) with exponent λ have been found when we reach an equation E_{m+1} that has no nonzero Laurent series solution.

To apply the original Frobenius or Heffter algorithm at an arbitrary singularity of a system of linear differential equations would require transforming the system to a scalar differential equation (*e.g.* by the cyclic vector method). That scalar equation usually has huge coefficients, making this approach quite unpractical. If $z = 0$ is a regular singularity of a first-order system of the form

$$\frac{dY}{dz} = A(z)Y(z), \quad A(z) \in \mathrm{Mat}_N(C(z)) \tag{2}$$

then, it is possible in theory to use a variant of the Frobenius algorithm that can be applied directly [8, p. 136, exercise 13], but this approach cannot be applied to irregular singularities or higher-order systems.

A generalization of Heffter's algorithm for constructing the regular solutions of first order systems of the form (2) is described in [5, §5] (an extended version is in [6, §3]). That algorithm is direct, *i.e.* it does not use any uncoupling procedure. A necessary step is however to find all the Laurent series solutions of a given system. For this task and for producing the indicial equation $f(\lambda) = 0$, the algorithm of [6,5] transforms the system into its *super-irreducible* form (see [11]).

We describe in this paper another adaptation of Heffter's algorithm to linear differential system, which uses the desingularization algorithm of [2,3] instead of transforming a system into its super-irreducible form. This allows us to handle higher order systems, *i.e.* operators such as (1) where the Q_i are matrices of polynomials, directly, *i.e.* without converting them to larger first-order systems. We study the efficiency of the approach both from the theoretical and practical viewpoints, and it shows that solving them directly is more efficient.

In a similar way, we solve the related problem of finding all the solutions with entries in $C(z)[\log z]$, which we call *logarithmic* solutions. This problem is decomposed into first finding a universal denominator for the solutions, and then finding solutions with entries in $C[z][\log z]$. The latter problem is solved by a slightly modified version of our algorithm for the regular solutions. For the denominator, in addtion, we propose a new hybrid method that combines the algorithm of [2] with a reduction algorithm specific to regular singularities [7]. The hybrid method is applicable to the case of first order systems, and in this case it speeds up the computation quite often (see Sect. 5).

2 Desingularization of Linear Recurrence Systems

Linear recurrences with variable coefficients are of interest for many applications (*e.g.* combinatorics and numeric computation). Consider a recurrence of the form

$$P_l(n)x_{n+l} + P_{l-1}(n)x_{n+l-1} + \cdots + P_t(n)x_{n+t} = r_n \tag{3}$$

where $l \geq t$ are arbitrary integers, $x = (x^1, \ldots, x^N)^T$ is a column vector of unknown sequences (such that $x_i = (x_i^1, \ldots, x_i^N)^T$), $P_t(n), \ldots, P_l(n) \in \mathrm{Mat}_N(C[n])$,

$P_t(n) \neq 0 \neq P_l(n)$ and $r_n \in C[n]^N$. The matrices $P_l(n)$ and $P_t(n)$ are called respectively the *leading* and *trailing* matrices of the recurrence. When $P_t(n)$ and $P_l(n)$ are nonsingular, the roots of their determinants are important for determining the structure of the solution space, as they give bounds on the solutions whose support is bounded above or below. It may happen however that $P_t(n)$ or $P_l(n)$ is singular (or both). In that case, they do not yield bounds on the solutions, but it is also difficult, from a computational standpoint, to use the recurrence (3) to compute the sequence of vectors that it generates. A natural solution in that case is to transform the recurrence system into an equivalent one with either the leading or trailing matrix nonsingular. That transformation may be a "quasi–equivalence", in the sense that the eventual changes in the solution space can be easily described. Such a transformation (the EG-algorithm) was developed in [1] and later improved in [2]. In addition to the transformed system, it also yields a finite set of linear constraints such that the solutions of the original system are exactly those of the transformed system that also satisfy the new constraints (each of the constraints is a linear relation that contains a finite set of variables x_i^j).

3 Regular Solutions

We consider in this section the higher order system $LY = 0$ where L is of the form (1) with $Q_0, \ldots, Q_\rho \in \mathrm{Mat}_N(C[z])$ and Q_ρ nonsingular.

3.1 Description of the Algorithm

Using the standard basis $(z^m)_{m\geq 0}$ of $C[z]$, we construct (see [2, §2]) its associated recurrence system $Rc = 0$, where $R = P_l(n)E^l + \cdots + P_t(n)E^t$, E is the shift operator and $P_j(n) \in \mathrm{Mat}_N(C[n])$ for $t \leq j \leq l$. If $\det P_l(n)$ is identically 0, then it is possible (see Section 2) to transform the recurrence system into an equivalent one (together with a finite set of linear constraints) with $\det P_l(n) \neq 0$, so assume from now on that $\varphi(n) = \det P_l(n) \neq 0$. Let $\psi(n) = \varphi(n-l)$ and n_0, n_1 be respectively the minimal and maximal integer roots of $\psi(n)$ (if there is no integer root, then $LY = 0$ has no Laurent series solution). Any Laurent series solution of $LY = 0$ has no term $c_k z^k$ with $c_k \in C^N$ and $k < n_0$. Using the recurrence $Rc = 0$ and the additional constraints, we can, by a linear algebra procedure, compute a basis of the linear space of initial segments $c_{n_0}z^{n_0} + c_{n_0+1}z^{n_0+1} + \cdots + c_M z^M$, where M is a fixed integer, chosen greater that n_1 and all the indices appearing in the linear constraints. Observe that if our differential system is inhomogeneous with a Laurent series right-hand side (whose coefficients are given by a linear recurrence system), then we can similarly construct a basis of the affine space of its Laurent series solutions. If $\psi(n)$ has a non-integer root λ, then the preliminary change of variable $Y = z^\lambda \bar{Y}$ produces a new system for $\bar{Y}$, hence a new recurrence with a new $\bar{\psi}(n) = \psi(n - \lambda)$. Therefore, we can always work with the integer roots of ψ. For any integer $m \geq 0$, the result of applying L to $g(z)\log^m(z)/m!$ is clearly of the form

$$L_{m,m}(g)\frac{\log^m z}{m!} + \cdots + L_{m,1}(g)\frac{\log z}{1!} + L_{m,0}(g), \quad (4)$$

where the coefficients of the differential operators $L_{i,j}$ belong to $\mathrm{Mat}_N(C(z))$. Proofs of the following proposition can be found in [10] and [12, Sect. 3.2.1].

Proposition 1. *The coefficients of all the $L_{i,j}$ in (4) belong to* $\mathrm{Mat}_N(C[z, z^{-1}])$. *In addition, $L_{0,0} = L$ and $L_{i+j,j} = L_{i,0}$ for any $i, j \geq 0$.*

Let $L_i = L_{i,0}(= L_{i+j,j}$ for any $j \geq 0)$. Using (4) and Proposition 1 we obtain

$$L\left(\sum_{m=0}^{k} g_{k-m}(z)\frac{\log^m z}{m!}\right) = \sum_{m=0}^{k}\left(\sum_{j=0}^{k-m} L_j(g_{k-m-j})\right)\frac{\log^m z}{m!}.$$

Therefore,

$$Y = \sum_{m=0}^{k} g_{k-m}(z)\frac{\log^m z}{m!} \tag{5}$$

is a solution of $LY = 0$ if and only if $(g_0(z), \ldots, g_k(z))$ is a Laurent series solution of the inhomogeneous linear system

$$L_0(g_i) = -\sum_{j=1}^{i} L_j(g_{i-j}) \quad \text{for } 0 \leq i \leq k\,. \tag{6}$$

When we find $g_0(z)$ using the first equation $L_0(g_0) = 0$ of (6), that solution contains arbitrary constants. When we use $g_0(z)$ in the right-hand side of the next equation $L_0(g_1) = -L_1(g_0)$ of (6) those arbitrary constants appear linearly in the right-hand side. Using the same technique as when solving such scalar parametric inhomogeneous equations (see for example [4]), we find together with $g_1(z)$ linear constraints on the arbitrary constants appearing in g_0 and g_1. Repeating this process, we find at each step that $g_0, \ldots, g_i$ depend on unknown constants together with a linear system for those constants. In order for this process to terminate, we need to ensure that we always reach an integer k such that (6) has no Laurent series solution with $g_0 \neq 0$. Heffter proved this in the scalar case, and his proof carries over to systems.

Proposition 2. *The set $K = \{k \geq 0$ such that (6) has a solution with $g_0 \neq 0\}$ is finite. If K is empty, then $LY = 0$ has no nonzero solution in $C((z))[\log z]$. Otherwise, $K = \{0, \ldots, \mu\}$ for some $\mu \geq 0$ and any solution in $C((z))[\log z]$ of $LY = 0$ has the form (5) where $(g_0, \ldots, g_\mu)$ is a solution of (6) with $k = \mu$. In addition, any solution of (6) with entries in $C((z))$ generates a solution of $LY = 0$.*

Proof. Let G_k be the linear space of all the (regular) solutions of the form (5) of $LY = 0$, and $Y \in G_k$. Writing $Y = \sum_{m=0}^{k+1} h_{k+1-m}(z) \log^m(z)/m!$ where $h_0 = 0$ and $h_{i+1} = g_i$ for $0 \leq i \leq k$, we see that $G_0 \subseteq G_1 \subseteq \cdots \subseteq G_k \subseteq G_{k+1} \subseteq \cdots$. Let $k > 0$ be in K and $(g_0, \ldots, g_k)$ be a solution of (6) with $g_0 \neq 0$. Then, $(g_0, \ldots, g_{k-1})$ is a solution of (6) with $k - 1$ and so on, which implies that $\{0, \ldots, k\} \subset K$ and that $\dim_C G_k \geq k$. This produces k linearly independent