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Alexander Prestel

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To the memory of my parents

Preface

Ten years ago, in 1974, these lectures were given at the 'Instituto de Matemática Pura e Aplicada', in Rio de Janeiro. In 1975 the notes of these lectures appeared in the series 'Monografias de Matemática' as No.22 published by IMPA. From that time on the notes served as an introductory text for the theory of real fields and its connections with valuation theory and quadratic form theory.

Since the 'Lectures on formally real fields' have been used in many publications as a standard reference and since currently they still seem to be the only introductory text to this theory, I gratefully accepted the proposal of C. Camacho to republish them in the IMPA-Subseries of the 'Lecture Notes in Mathematics'. It seems wise not to make changes in this new edition apart from correcting misprints and a few minor errors: The only change made is to replace the term q-*ordering* by *semiordering*, since the latter term turned out to be almost exclusively used in the recent literature.

For the developments during the last decade in this theory, particularly in the theory of reduced quadratic forms, I would like to refer the reader to two publications of T.Y. Lam: his survey article

The Theory of Ordered Fields [in Ring Theory and Algebra III (ed. B. Mc Donald), Lecture Notes in Pure and Applied Math., Vol. 55, Dekker, New York, 1980, p. 1-152] and to his expository notes

Orderings, Valuations and Quadratic Forms [Conf. Board of the Math. Sciences, Regional Conf. Series in Math., No. 52, Providence,R.I.,1983].

Finally I would like to thank Edda Polte for preparing the typescript of this second edition.

Konstanz, 1984 A. Prestel

Introduction

In mathematics, the following method of generalization has turned out to be very successful: one starts with a given well-known mathematical structure, singles out the most basic properties of that structure (axioms) and then considers the class of all structures satisfying these properties. Finally, one tries to characterize the original structure among the members of this class.

A well known example of that method is the following: considering the set $\mathbb{C}$ of complex numbers together with the usual (field-) operations, the most basic properties are the rules for dealing with these operations. All these rules are implied by the "axioms" of fields of characteristic zero. Among this class of fields, $\mathbb{C}$ cannot be characterized completely using algebraic properties only. But there is a subclass consisting of the algebraically closed fields of characteristic zero which contains $\mathbb{C}$ and shares with $\mathbb{C}$ all "algebraic properties". This fact is usually called the "Lefschetz-Principle".

Besides $\mathbb{C}$, the field $\mathbb{R}$ of real numbers is another outstanding mathematical structure. One of the main differences between $\mathbb{C}$ and $\mathbb{R}$ is the existence of a so-called "ordering" on $\mathbb{R}$. Now, the most basic properties are the field axioms together with the rules for dealing with the ordering "$\leq$". These rules are all implied by the axioms of a linear order and

(1) $x \leq y \quad \Rightarrow \quad x + z \leq y + z$

(2) $x \leq y,\ 0 \leq z \quad \Rightarrow \quad xz \leq yz$.

A field F together with a linear ordering $\leq$ satisfying (1) and (2) is called an ordered field $(F,\leq)$. Among the class of ordered fields, $\mathbb{R}$ cannot be characterized completely using only algebraic and order-theoretical properties, without involving set-theoretical properties (completeness!). But once more, there is a subclass, the class of maximally ordered (or real closed) fields, which shares with $\mathbb{R}$ all algebraic and order-theoretical properties. This fact is usually called the "Tarski-Principle".

All this is contained in § 1 to § 5. To state these "Transfer-Principles" precisely and to prove them, we need some notions from model theory (introduced in § 4). The theory presented is self-contained up to the proof of the main theorem of model theory, the compactness theorem.

A field F admitting at least one ordering, in general admits several. The set X_F of all orderings of F can be given a topology in such a way that it becomes a compact and totally disconnected space. The study of such spaces is contained in § 6 to § 9.

A field admitting at least one ordering is called <u>formally real</u>. These fields are characterized algebraically by the condition: -1 is not a sum of squares in F , or equivalently, no quadratic equation of the type

$$X_1^2 + \ldots + X_n^2 = 0$$

has a non-trivial solution in F . This characterization is one of the most elementary connections between the theory of formally real fields and the theory of quadratic forms. More generally, if $\leq$ is an ordering of the field F and $a_1, \ldots, a_n \in F$ are positive with respect to $\leq$, then the quadratic equation

$$(*) \qquad a_1 X_1^2 + \ldots + a_n X_n^2 = 0$$

admits no non-trivial solution in F . This is based on the important consequence

$$(2') \qquad 0 \leq x \Rightarrow 0 \leq xy^2$$

of (2). Hence, even if $a_1,\dots,a_n > 0$ and $\leq$ is a <u>semiordering</u> of F, i.e. a linear ordering satisfying (1), (2') and $0 < 1$, then (*) admits no non-trivial solution in F.

This observation indicates the usefulness of the study of semiorderings in connection with quadratic forms over formally real fields. Hence, right from the beginning we deal with semiorderings. These turn out to be of great importance in proving certain "Local-Global Principles" (see § 8).

One class of fields turns out to be of special interest. It consists of all fields in which every semiordering is already an ordering. The space X_F of orderings in this case satisfies the so-called "Strong Approximation Property". Hence these fields are called SAP-fields. This class will be studied in § 9. In § 10 SAP-fields will be characterized by their Witt rings of quadratic forms.

These notes are based on a course on formally real fields taught at IMPA during the winter period from April to June 1974. The author's stay at IMPA took place under the German-Brazilian Cooperation Agreement GMD-CNPq. The typescript was prepared by Wilson Góes. I want to thank him for the fine job he did.

Rio de Janeiro 1975 A. Prestel

CONTENTS

§ 1. ORDERINGS AND SEMIORDERINGS OF FIELDS

Orderings

Let F be a field. An ordering $\leq$ of F is a binary relation satisfying

(1.1) (i) $a \leq a$

(ii) $a \leq b,\ b \leq c \Rightarrow a \leq c$

(iii) $a \leq b,\ b \leq a \Rightarrow a = b$

(iv) $a \leq b$ or $b \leq a$

(v) $a \leq b \Rightarrow a + c \leq b + c$

(vi) $0 \leq a,\ 0 \leq b \Rightarrow 0 \leq ab$.

The set $P = \{a \in F \mid 0 \leq a\}$ obviously satisfies

(1.2) (1) $P + P \subset P$

(2) $P \cdot P \subset P$

(3) $P \cap -P = \{0\}$

(4) $P \cup -P = F$.

P is called the positive cone of $\leq$ (although it includes 0). A subset $P \subset F$ satisfying (1) to (4) is called a positive cone of F . It is easy to see that, for a positive cone P,

$$a \leq b :\Longleftrightarrow b - a \in P$$

defines an ordering on F such that P is its positive cone. Hence we will sometimes call P an ordering as well. Examples of ordered fields are $\mathbb{Q}$ and $\mathbb{R}$ with their usual orderings.

To get an algebraic criterion for fields which admit some ordering, we generalize the notion of positive cones to pre-positive cones of fields. But looking forward to §5 we will go one step further and also consider rings.

Let A be a commutative ring with unit 1 and I a subset of A.

Then we call a subset P of A a <u>pre-positive cone of</u> A if

(1.3) (1) $P + P \subset P$

(2) $P \cdot P \subset P$

(3) $-1 \notin P$

(4) $A^2 \subset P$.

(1.4) LEMMA *Let* P_o *be a pre-positive cone of* A . *Then there is an extension* P *of* P_o *satisfying in addition:*

$$P \cup -P = A \quad \text{and} \quad P \cap -P \quad \text{is a prime ideal.}$$

Proof: By Zorn's lemma the set of pre-positive cones extending P_o contains some maximal element. Let $P \supset P_o$ be such a maximal pre-positive cone. We claim

$$Px \cap (1+P) = \emptyset \quad \text{or} \quad -Px \cap (1+P) = \emptyset$$

for all $x \in A$. Suppose we have $p_1 x = 1 + q_1$ and $-p_2 x = 1 + q_2$ for some $p_1, p_2, q_1, q_2 \in P$. Multiplying both equations we obtain $-p_1 p_2 x^2 \in 1 + P$. But then $-1 \in P$ which is a contradiction to (3).

To prove $P \cup -P = A$ let $x \in A$ and assume first $Px \cap (1+P) = \emptyset$. For $P' := P - Px$ we get $P \subset P'$, $-x \in P'$, $P' + P' \subset P'$ and $P' \cdot P' \subset P'$. If $-1 \in P' = P - Px$ then $Px \cap (1+P) \neq \emptyset$. Hence P' is again a pre-positive cone. But by the maximality of P, $P' = P$. Hence $-x \in P$. If we assume now $-Px \cap (1+P) = \emptyset$ we get $x \in P$ by the same argument. It remains to show that $J := P \cap -P$ is a prime ideal of A . From $P \cup -P = A$ it follows easily that J is an ideal. Now let $a_1 \cdot a_2 \in J$. We may assume without loss of generality $-a_1, -a_2 \notin P$. But then from the above argument we get $Pa_1 \cap (1+P) \neq \emptyset$ and $Pa_2 \cap (1+P) \neq \emptyset$. Let $p_1 a_1 = 1 + q_1$ and $p_2 a_2 = 1 + q_2$ for some $p_1, p_2, q_1, q_2 \in P$. Multiplying we obtain $p_1 p_2 a_1 a_2 \in 1 + P$. Hence $-1 \in P - p_1 p_2 a_1 a_2 \subset P + J \subset P$ gives a contradiction.

q.e.d.

In case A is a field F, we call $P \subset F$ a <u>pre-positive cone</u> of F if it satisfies (1.3) (1) to (4). Obviously any positive cone P of F is also a pre-positive cone, since for any $x \in F$ we have $x \in P$ and hence $x^2 \in P$ or $-x \in P$ and hence $x^2 = (-x)(-x) \in P$. In particular we have $1^2 = 1 \in P$, which implies $-1 \notin P$.

(1.5) COROLLARY <u>Any pre-positive cone</u> P_o <u>of a field</u> F <u>can be extended to some positive cone</u> P <u>of</u> F.

From the proof of Lemma (1.4) we get another

(1.6) COROLLARY <u>Any pre-positive cone</u> P_o <u>of a field</u> F <u>equals the intersection of all positive cones</u> P <u>extending</u> P_o.

<u>Proof</u>: Obviously P_o is included in the intersection. Now let $x \notin P_o$. But then $P_o x \cap (1+P_o) = \emptyset$, since from $px = 1 + q$ and $p,q \in P_o$ we would get $x = (1+q)\, p(\frac{1}{p})^2 \in P_o$. As in the proof of (1.4), $P' = P_o - x P_o$ is a pre-positive cone extending P_o and containing $-x$. By Corollary (1.5), P' can be extended to a positive cone of F. Hence x is not in the intersection.

q.e.d.

Now let

$$S_F = \text{set of sums of squares of } F .$$

(1.7) PROPOSITION (a) S_F <u>is contained in every pre-positive cone of</u> F.

(b) S_F <u>is closed under addition</u>.

(c) $\dot{S}_F$ <u>is a multiplicative subgroup of</u> $\dot{F}$.

<u>Proof</u>: (a) and (b) are trivial.

(c) $$\Sigma a_i^2 \neq 0 \Rightarrow (\Sigma a_i^2)^{-1} = \Sigma \left(\frac{a_i}{\Sigma a_i^2}\right)^2 .$$

q.e.d.

The following theorem gives an algebraic characterization of fields which admit some ordering. These fields will be called <u>formally real</u> or <u>orderable</u>.

(1.8) THEOREM <u>For a field</u> F , (a) <u>to</u> (d) <u>are equivalent</u>:

(a) F <u>is formally real</u>

(b) $-1 \notin S_F$

(c) $\Sigma a_i^2 = 0 \Rightarrow$ <u>all</u> $a_i = 0$

(d) $F \neq S_F$.

<u>Proof</u>: (b) $\Leftrightarrow$ (c) and (a) $\Rightarrow$ (d) are easy to see.

(d) $\Rightarrow$ (b): If $-1 \in S_F$, by $a = (\frac{a+1}{2})^2 + (-1)(\frac{a-1}{2})^2$, $a \in S_F$ for any $a \in F$, i.e. $S_F = F$.

(b) $\Rightarrow$ (a): If $-1 \notin S_F$, S_F is a pre-positive cone of F , hence by Corollary (1.5) can be extended to a positive cone of F .

q.e.d.

(1.9) COROLLARY $S_F = \bigcap_{P \text{ positive cone}} P$.

<u>Proof</u>: If F is not orderable, $S_F = F$ and the intersection over the empty index set is by definition F . If F is orderable, this equation is a special case of Corollary (1.6).

q.e.d.

Hence the elements of S_F are also called <u>totally positive</u>.

<u>Remark</u>: The characteristic of a formally real field F is obviously zero. Hence the field $\mathbb{Q}$ of rational numbers is isomorphic to a subfield of F .

(1.10) PROPOSITION <u>Let</u> P_1, P_2 <u>be positive cones of a field</u> F . <u>Then</u> $P_1 \subset P_2$ <u>implies</u> $P_1 = P_2$.

Proof: Let $x \in P_2$ and $x \notin P_1$. But then $-x \in P_1 \subset P_2$ gives a contradiction.

q.e.d.

(1.11) LEMMA S_F is a positive cone of a field F iff F has a unique ordering.

Proof: "$\Rightarrow$" by Proposition (1.10).

"$\Leftarrow$" suppose $S_F \cup -S_F \neq F$. Let $x, -x \notin S_F$. Then by Corollary (1.9) there are positive cones P_1, P_2 $x \notin P_1$, $-x \notin P_2$. But then $-x \in P_1$, $x \in P_2$ gives a contradiction.

q.e.d.

Semiorderings

As we remarked in the introduction there is a certain generalization of positive cones and orderings which is based on the following observation. Very often one only uses the property

$$0 \leq a \Rightarrow 0 \leq ab^2$$

of an ordering together with $0 < 1$. This is especially the case if one deals with quadratic forms. Hence we will call a binary relation $\leq$ of a field F a semiordering if it satisfies (1.1) (i) to (v) and

(vi!) $0 < 1$

$$0 \leq a \Rightarrow 0 \leq ab^2$$

for all $a,b \in F$. The set $P = \{a \in P \mid 0 \leq a\}$ then satisfies

(1) $P + P \subset P$

(2) $F^2 \cdot P \subset P$ and $1 \in P$

(3) $P \cap -P = 0$

(4) $P \cup -P = F$.

Such a set will be called a <u>semicone</u> of F . Obviously a semicone $P \subset F$ determines a semiordering $\leq$ of F defined by

$$a \leq b :\iff b - a \in P .$$

Hence we will sometimes call P a semiordering as well.

Any ordering of F is also a semiordering. But there are semiorderings $\leq$ of some fields F which are not orderings (as will be seen in §6), i.e. there are $a,b \in F$ such that $0 < a,b$ and $ab < 0$. A semiordering which is not an ordering will be called a <u>proper semiordering</u>.

For semiorderings the statements corresponding to (1.5) to (1.11) also hold. To see this let us first introduce the notion of a pre-semicone, which is not quite analogous to that of a pre-positive cone. We call a subset $P \subset F$ a <u>pre-semicone</u> if it satisfies

(1) $P + P \subset P$

(2) $F^2 \cdot P \subset P$

(3) $P \cap -P = \{0\}$.

Nothing is said about 1. Obviously any semicone is a pre-semicone. (2) may be replaced by $S_F \cdot P \subset P$.

(1.12) LEMMA If P is a <u>pre-semicone of a formally real field</u> F <u>and</u> $x \notin P$ <u>then there is a pre-semicone</u> P' <u>extending</u> P <u>and containing</u> $-x$.

<u>Proof</u>: Let $P' = P - x S_F$. Obviously P' satisfies (1) and (2). Let $p - xs \in -P'$ for some $p \in P$, $s \in S_F$. Then there are $p_1 \in P$, $s_1 \in S_F$ such that $(p+p_1) - x(s+s_1) = 0$. If $s + s_1 \neq 0$, $x = (p+p_1)(s+s_1)^{-1} \in P$ gives a contradiction. Hence $s + s_1 = 0$. Since F is formally real, $s = s_1 = 0$ by (1.8) (c). Hence $p + p_1 = 0$. But then $p \in P \cap -P$, which implies $p = p_1 = 0$.

q.e.d.

(1.13) LEMMA <u>Any pre-semicone</u> P_o <u>of a formally real[1] field</u> F <u>can be extended to some</u> P <u>such that</u> P <u>or</u> $-P$ <u>is a semicone of</u> F.

<u>Proof</u>: By Zorn's lemma the set of pre-semicones extending P_o contains some maximal pre-semicone P. If then $x \notin P$, by Lemma (1.12) there is an extension P' of P such that $-x \in P'$. From the maximality of P we get $-x \in P$. Hence $P \cup -P = F$. Now P or $-P$ is a semicone.

q.e.d.

(1.14) COROLLARY <u>If</u> F <u>is formally real</u>, S_F <u>equals the intersection of all semicones</u>.

<u>Proof</u>: Follows directly from (1.12) and (1.13). Note that S_F is a pre-semicone containing 1 if F is formally real.

q.e.d.

(1.15) COROLLARY F <u>is formally real iff it admits a semiordering</u>.

By the same argument as in (1.10) we get:

(1.16) PROPOSITION <u>If</u> $P_1 \subset P_2$ <u>are both semicones of</u> F, <u>then</u> $P_1 = P_2$.

(1.17) COROLLARY <u>If</u> F <u>has a unique (semi)ordering, it has no proper semiordering</u>.

<u>Proof</u>: By the same argument as in (1.11) the assumption implies that S_F forms a positive cone of F. Now any semicone P contains S_F. Hence $P = S_F$ by (1.16).

q.e.d.

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[1] Actually "formally real" is superfluous. For a pre-semicone P_o, $S_F \subset \pm P_o$ and hence $-1 \notin S_F$.