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Means of Hilbert Space Operators

$$M(H,K)X = \int_0^{\|H\|}\int_0^{\|K\|} M(s,t)\, dE_s X\, dF_t$$

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Preface

Roughly speaking two kinds of operator and/or matrix inequalities are known, of course with many important exceptions. Operators admit several natural notions of orders (such as positive semidefiniteness order, some majorization orders and so on) due to their non-commutativity, and some operator inequalities clarify these order relations. There is also another kind of operator inequalities comparing or estimating various quantities (such as norms, traces, determinants and so on) naturally attached to operators.

Both kinds are of fundamental importance in many branches of mathematical analysis, but are also sometimes highly non-trivial because of the non-commutativity of the operators involved. This monograph is mainly devoted to means of Hilbert space operators and their general properties with the main emphasis on their norm comparison results. Therefore, our operator inequalities here are basically of the second kind. However, they are not free from the first in the sense that our general theory on means relies heavily on a certain order for operators (i.e., a majorization technique which is relevant for dealing with unitarily invariant norms).

In recent years many norm inequalities on operator means have been investigated. We develop here a general theory which enables us to treat them in a unified and axiomatic fashion. More precisely, we associate operator means to given scalar means by making use of the theory of Stieltjes double integral transformations. Here, Peller's characterization of Schur multipliers plays an important role, and indeed guarantees that our operator means are bounded operators. Basic properties on these operator means (such as the convergence property and norm bounds) are studied. We also obtain a handy criterion (in terms of the Fourier transformation) to check the validity of norm comparison among operator means.

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Contents

1

Introduction

The present monograph is devoted to a thorough study of means for Hilbert space operators, especially comparison of (unitarily invariant) norms of operator means and their convergence properties in various aspects.

The Hadamard product (or Schur product) $A \circ B$ of two matrices $A = [a_{ij}]$, $B = [b_{ij}]$ means their entry-wise product $[a_{ij}b_{ij}]$. This notion is a common and powerful technique in investigation of general matrix (and/or operator) norm inequalities, and particularly so in that of perturbation inequalities and commutator estimates. Assume that $n \times n$ matrices $H, K, X \in M_n(\mathbf{C})$ are given with $H, K \geq 0$ and diagonalizations

$$H = U\mathrm{diag}(s_1, s_2, \ldots, s_n)U^* \text{ and } K = V\mathrm{diag}(t_1, t_2, \ldots, t_n)V^*.$$

In our previous work [39], to a given scalar mean $M(s, t)$ (for $s, t \in \mathbf{R}_+$), we associated the corresponding matrix mean $M(H, K)X$ by

$$M(H, K)X = U\left([M(s_i, t_j)] \circ (U^*XV)\right)V^*. \tag{1.1}$$

For a scalar mean $M(s, t)$ of the form $\sum_{i=1}^{n} f_i(s)g_i(t)$ one easily observes $M(H, K)X = \sum_{i=1}^{n} f_i(H)Xg_i(K)$, and we note that this expression makes a perfect sense even for Hilbert space operators H, K, X with $H, K \geq 0$. However, for the definition of general matrix means $M(H, K)X$ (such as A-L-G interpolation means $M_\alpha(H, K)X$ and binomial means $B_\alpha(H, K)X$ to be explained later) the use of Hadamard products or something alike seems unavoidable.

The first main purpose of the present monograph is to develop a reasonable theory of means for Hilbert space operators, which works equally well for general scalar means (including M_α, B_α and so on). Here two difficulties have to be resolved: (i) Given (infinite-dimensional) diagonal operators $H, K \geq 0$, the definition (1.1) remains legitimate for $X \in \mathcal{C}_2(\mathcal{H})$, the Hilbert-Schmidt class operators on a Hilbert space $\mathcal{H}$, as long as entries $M(s_i, t_j)$ stay bounded (and $M(H, K)X \in \mathcal{C}_2(\mathcal{H})$). However, what we want is a mean $M(H, K)X$ ($\in B(\mathcal{H})$) for each bounded operator $X \in B(\mathcal{H})$. (ii) General positive operators H, K are no longer diagonal so that continuous spectral decomposition has to be used. The requirement in (i) says that the concept of a Schur multiplier ([31, 32, 66]) has to enter our picture, and hence what we need is a continuous analogue of the operation (1.1) with this concept built in. The theory of (Stieltjes) double integral transformations ([14]) due to M. Sh. Birman, M. Z. Solomyak and others is suited for this purpose. With this apparatus the operator mean $M(H, K)X$ is defined (in Chapter 3) as

$$M(H, K)X = \int_0^{\|H\|} \int_0^{\|K\|} M(s, t) \, dE_s X dF_t \tag{1.2}$$

with the spectral decompositions

$$H = \int_0^{\|H\|} s \, dE_s \quad \text{and} \quad K = \int_0^{\|K\|} t \, dF_t.$$

Double integral transformations as above were actually considered with general functions $M(s, t)$ (which are not necessarily means). This subject has important applications to theories of perturbation, Volterra operators, Hankel operators and so on (see §2.5 for more information including references), and one of central problems here (besides the justification of the double integral (1.2)) is to determine for which unitarily invariant norm the transformation $X \mapsto M(H, K)X$ is bounded. Extensive study has been made in this direction, and V. V. Peller's work ([69, 70]) deserves special mentioning. Namely, he completely characterized ($\mathcal{C}_1$-)Schur multipliers in this setting (i.e., boundedness criterion relative to the trace norm $\|\cdot\|_1$, or equivalently, the operator norm $\|\cdot\|$ by the duality), which is a continuous counterpart of U. Haagerup's characterization ([31, 32]) in the matrix setting. Our theory of operator means is built upon V. V. Peller's characterization (Theorem 2.2) although just an easy part is needed. Unfortunately, his work [69] with a proof (while [70] is an announcement) was not widely circulated, and details of some parts were omitted. Moreover, quite a few references there are not easily accessible. For these reasons and to make the monograph as self-contained as possible, we present details of his proof in Chapter 2 (see §2.1).

As emphasized above, the notions of Hadamard products and double integral transformations play important roles in perturbation theory and commutator estimates. In this monograph we restrict ourselves mainly to symmetric homogeneous means (except in Chapter 8 and §A.1) so that these important topics will not be touched. However, most of the arguments in Chapters 2 and 3 are quite general and our technique can be applicable to these topics (which will be actually carried out in our forthcoming article [55]). It is needless to say that there are large numbers of literature on matrix and/or operator norm inequalities (not necessarily of perturbation and/or commutator-type) based on closely related techniques. We also remark that the technique here is useful for dealing with certain operator equations such as Lyapunov-type equations (see §3.7 and [39, §4]). These related topics as well as relationship to other standard methods for study of operator inequalities (such as majorization theory and so on) are summarized at the end of each chapter together with suitable references, which might be of some help to the reader.

In the rest we will explain historical background at first and then more details on the contents of the present monograph. In the classical work [36] E. Heinz showed the (operator) norm inequality

$$\|H^{\theta}XK^{1-\theta}+H^{1-\theta}XK^{\theta}\| \le \|HX+XK\| \quad (\text{for } \theta\in[0,1]) \tag{1.3}$$

for positive operators $H, K \ge 0$ and an arbitrary operator X on a Hilbert space. In the 1979 article [64] A. McIntosh presented a simple proof of

$$\|H^*XK\| \le \frac{1}{2}\|HH^*X+XKK^*\|,$$

which is obviously equivalent to the following estimate for positive operators:

$$\|H^{1/2}XK^{1/2}\| \le \frac{1}{2}\|HX+XK\| \quad (H,K\ge 0).$$

It is the special case $\theta = 1/2$ of (1.3), and he pointed out that a simple and unified approach to so-called Heinz-type inequalities such as (1.3) (and the "difference version" (8.7)) is possible based on this arithmetic-geometric mean inequality. The closely related eigenvalue estimate

$$\mu_n(H^{1/2}K^{1/2}) \le \frac{1}{2}\mu_n(H+K) \quad (n=1,2,\dots)$$

for positive matrices is known ([12]). Here, $\{\mu_n(\cdot)\}_{n=1,2,\cdots}$ denotes singular numbers, i.e., $\mu_n(Y)$ is the n-th largest eigenvalue (with multiplicities counted) of the positive part $|Y| = (Y^*Y)^{1/2}$. This means $|H^{1/2}K^{1/2}| \le \frac{1}{2}U(H+K)U^*$ for some unitary matrix U so that we have

$$|||H^{1/2}K^{1/2}||| \le \frac{1}{2}|||H+K|||$$

for an arbitrary unitarily invariant norm $|||\cdot|||$.

In the 1993 article [10] R. Bhatia and C. Davis showed the following strengthening:

$$|||H^{1/2}XK^{1/2}||| \le \frac{1}{2}|||HX+XK||| \tag{1.4}$$

for matrices, which of course remains valid for Hilbert space operators $H, K \ge 0$ and X by the standard approximation argument. On the other hand, in [3] T. Ando obtained the matrix Young inequality

$$\mu_n\left(H^{\frac{1}{p}}K^{\frac{1}{q}}\right) \le \mu_n\left(\tfrac{1}{p}H+\tfrac{1}{q}K\right) \quad (n=1,2,\dots) \tag{1.5}$$

for $p, q > 1$ with $p^{-1}+q^{-1}=1$. Although the weak matrix Young inequality

$$|||H^{\frac{1}{p}}XK^{\frac{1}{q}}||| \le \kappa_p|||\tfrac{1}{p}HX + \tfrac{1}{q}XK||| \tag{1.6}$$

holds with some constant $\kappa_p \ge 1$ ([54]), without this constant the inequality fails to hold for the operator norm $|||\cdot||| = \|\cdot\|$ (unless $p = 2$) as was pointed out in [2]. Instead, the following slightly weaker inequality holds always:

$$|||H^{\frac{1}{p}}XK^{\frac{1}{q}}||| \le \tfrac{1}{p}|||HX||| + \tfrac{1}{q}|||XK|||. \tag{1.7}$$

In the recent years the above-mentioned arithmetic-geometric mean and related inequalities have been under active investigation by several authors, and very readable accounts on this subject can be found in [2, 8, 84]. Motivated by these works, in a series of recent articles [54, 38, 39] we have investigated simple unified proofs for known (as well as many new) norm inequalities in a similar nature, and our investigation is summarized in the recent survey article [40]. We also point out that closely related analysis was made in the recent article [13] by R. Bhatia and K. Parthasarathy. For example as a refinement of (1.4) the arithmetic-logarithmic-geometric mean inequality

$$|||H^{1/2}XK^{1/2}||| \le |||\int_0^1 H^x XK^{1-x}dx||| \le \frac{1}{2}|||HX + XK||| \tag{1.8}$$

was obtained in [38]. The technique in this article actually permitted us to compare these quantities with

$$|||\frac{1}{m}\sum_{k=1}^{m} H^{\frac{k}{m+1}}XK^{\frac{m+1-k}{m+1}}|||, \quad |||\frac{1}{n}\sum_{k=0}^{n-1} H^{\frac{k}{n-1}}XK^{\frac{n-1-k}{n-1}}|||, \tag{1.9}$$

and moreover in the appendix to [38] we discussed the $|||\cdot|||$-convergence

$$\begin{cases} \dfrac{1}{m}\displaystyle\sum_{k=1}^{m} H^{\frac{k}{m+1}}XK^{\frac{m+1-k}{m+1}} \to \int_0^1 H^x XK^{1-x}dx \quad (\text{as } m \to \infty), \\ \dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1} H^{\frac{k}{n-1}}XK^{\frac{n-1-k}{n-1}} \to \int_0^1 H^x XK^{1-x}dx \quad (\text{as } n \to \infty) \end{cases} \tag{1.10}$$

under certain circumstances.

The starting point of the analysis made in [39] was an axiomatic treatment on matrix means (i.e., matrix means $M(H,K)X$ (see (1.1)) associated to scalar means $M(s,t)$ satisfying certain axioms), and a variety of generalizations of the norm inequalities explained so far were obtained as applications. As in [39] a certain class of symmetric homogeneous (scalar) means is considered in the present monograph, but our main concern here is a study of corresponding means for Hilbert space operators instead. In order to be able to define $M(H,K)X$ ($\in B(\mathcal{H})$) for each $X \in B(\mathcal{H})$ (by the double integral transformation (1.2)), our mean $M(s,t)$ has to be a Schur multiplier in addition. For two such means $M(s,t), N(s,t)$ we introduce the partial order:

$M \preceq N$ if and only if $M(e^x,1)/N(e^x,1)$ is positive definite. If this is the case, then for non-singular positive operators H, K we have the integral expression

$$M(H,K)X = \int_{-\infty}^{\infty} H^{ix}(N(H,K)X)K^{-ix}d\nu(x) \tag{1.11}$$

with a probability measure ν (see Theorems 3.4 and 3.7 for the precise statement), and of course the Bochner theorem is behind. Under such circumstances (thanks to the general fact explained in §A.2) we actually have

$$|||M(H,K)X||| \le |||N(H,K)X||| \tag{1.12}$$

(even without the non-singularity of $H, K \ge 0$). This inequality actually characterizes the order $M \preceq N$, and is a source for a variety of concrete norm inequalities (as was demonstrated in [40]). The order $\preceq$ and (1.11), (1.12) were also used in [39] for matrices, but much more involved arguments are required for Hilbert space operators, which will be carried out in Chapter 3. It is sometimes not an easy task to determine if a given mean $M(s,t)$ is a Schur multiplier. However, the mean $M_\infty(s,t) = \max\{s,t\}$ comes to the rescue: (i) The mean M_∞ itself is a Schur multiplier. (ii) A mean majorized by M_∞ (relative to $\preceq$) is a Schur multiplier. These are consequences of (1.11), (1.12), and enable us to prove that all the means considered in [39] are indeed Schur multipliers. The observation (i) also follows from the discrete decomposition of $\max\{s,t\}$ worked out in §A.3, which might be of independent interest. Furthermore, a general norm estimate of the transformation $X \mapsto M(H,K)X$ is established for means $M \preceq M_\infty$. In Chapter 4 we study the convergence $M(H_n,K_n)X \to M(H,K)X$ (in $|||\cdot|||$ or in the strong operator topology) under the strong convergence $H_n \to H$, $K_n \to K$ of the positive operators involved.

The requirement for the convergence (1.10) in the appendix to [39] was the following finiteness condition: either $|||H|||, |||K||| < \infty$ or $|||X||| < \infty$. This requirement is somewhat artificial (and too restrictive), and the arguments presented there were ad hoc. The second main purpose of the monograph is to present systematic and thorough investigation on such convergence phenomena. In [39] we dealt with the following one-parameter families of scalar means:

$$\begin{aligned}
M_\alpha(s,t) &= \frac{\alpha-1}{\alpha} \times \frac{s^\alpha - t^\alpha}{s^{\alpha-1}-t^{\alpha-1}} \quad (-\infty \le \alpha \le \infty),\\
A_\alpha(s,t) &= \frac{1}{2}(s^\alpha t^{1-\alpha} + s^{1-\alpha}t^\alpha) \quad (0 \le \alpha \le 1),\\
B_\alpha(s,t) &= \left(\frac{s^\alpha + t^\alpha}{2}\right)^{1/\alpha} \quad (-\infty \le \alpha \le \infty).
\end{aligned}$$

It is straight-forward to see that $M_\alpha(s,t), A_\alpha(s,t)$ are Schur multipliers, and also so is $B_{1/n}(s,t)$ thanks to the the binomial expansion $B_{1/n}(s,t) =$

$2^{-n}\sum_{k=0}^{n}\binom{n}{k}s^{\frac{k}{n}}t^{\frac{n-k}{n}}$. We indeed show that all of $B_\alpha(s,t)$ are (by proving $B_\alpha \preceq M_\infty$). Thus, all of the above give rise to operator means. Note $M_{1/2}(s,t)=\sqrt{st}$ (the geometric mean), $M_2=\frac{1}{2}(s+t)$ (the arithmetic mean) and

$$\begin{aligned} M_1(s,t) &\left(=\lim_{\alpha\to 1} M_\alpha(s,t)\right) \\ &= \frac{s-t}{\log s-\log t} = \int_0^1 s^x t^{1-x}\,dx \quad \text{(the logarithmic mean).} \end{aligned}$$

Because of these reasons $\{M_\alpha(s,t)\}_{-\infty\le\alpha\le\infty}$ will be referred to as the *A-L-G* interpolation means. The convergence (1.10) (see also (5.1)) means

$$\lim_{m\to\infty} |||M_{\frac{m}{m+1}}(H,K)X - L||| = \lim_{n\to\infty} |||M_{\frac{n}{n-1}}(H,K)X - L||| = 0$$

with the logarithmic mean $L=M_1(H,K)X=\int_0^1 H^xXK^{1-x}dx$, and the main result in Chapter 5 is the following generalization:

$$\lim_{\alpha\to\alpha_0} |||M_\alpha(H,K)X - M_{\alpha_0}(H,K)X||| = 0$$

under the assumption $|||M_\beta(H,K)X|||<\infty$ for some $\beta>\alpha_0$. This is a "dominated convergence theorem" for the *A-L-G* means, the proof of which is indeed based on Lebesgue's theorem applied to the relevant integral expression (1.11) with the concrete form of the density $d\nu(x)/dx$. Similar dominated convergence theorems for the Heinz-type means $A_\alpha(H,K)X = \frac{1}{2}(H^\alpha XK^{1-\alpha}+H^{1-\alpha}XK^\alpha)$ (or rather the single components $H^\alpha XK^{1-\alpha}$) and the binomial means $B_\alpha(H,K)X$ are also obtained together with other related results in Chapters 6 and 7.

A slightly different subject is covered in Chapter 8, that might be of independent interest. The homogeneous alternating sums

$$\begin{cases} \mathbf{A}(n) = \displaystyle\sum_{k=1}^{n}(-1)^{k-1}H^{\frac{k}{n+1}}XK^{\frac{n+1-k}{n+1}} & \text{(with } n=1,2,\cdots), \\ \mathbf{B}(m) = \displaystyle\sum_{k=0}^{m-1}(-1)^{k}H^{\frac{k}{m-1}}XK^{\frac{m-1-k}{m-1}} & \text{(with } m=2,3,\cdots) \end{cases}$$

are not necessarily symmetric (depending upon parities of n,m), but our method works and integral expressions akin to (1.11) (sometimes with signed measures ν) are available. This enables us to determine behavior of unitarily invariant norms of these alternating sums of operators such as mutual comparison, uniform bounds, monotonicity and so on.

Some technical results used in the monograph are collected in Appendices, and §A.1 is concerned with extension of our arguments to certain non-symmetric means.

2

Double integral transformations

Throughout the monograph a Hilbert space $\mathcal{H}$ is assumed to be separable. The algebra $B(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ is a Banach space with the operator norm $\|\cdot\|$. For $1 \le p < \infty$ let $\mathcal{C}_p(\mathcal{H})$ denote the Schatten p-class consisting of (compact) operators $X \in B(\mathcal{H})$ satisfying $\mathrm{Tr}(|X|^p) < \infty$ with $|X| = (X^*X)^{1/2}$, where Tr is the usual trace. The space $\mathcal{C}_p(\mathcal{H})$ is an ideal of $B(\mathcal{H})$ and a Banach space with the Schatten p-norm $\|X\|_p = (\mathrm{Tr}(|X|^p))^{1/p}$. In particular, $\mathcal{C}_1(\mathcal{H})$ is the trace class, and $\mathcal{C}_2(\mathcal{H})$ is the Hilbert-Schmidt class which is a Hilbert space with the inner product $(X,Y)_{\mathcal{C}_2(\mathcal{H})} = \mathrm{Tr}(XY^*)$ $(X, Y \in \mathcal{C}_2(\mathcal{H}))$. The algebra $B(\mathcal{H})$ is faithfully (hence isometrically) represented on the Hilbert space $\mathcal{C}_2(\mathcal{H})$ by the left (also right) multiplication: $X \in \mathcal{C}_2(\mathcal{H}) \mapsto AX, XA \in \mathcal{C}_2(\mathcal{H})$ for $A \in B(\mathcal{H})$. Standard references on these basic topics (as well as unitarily invariant norms) are [29, 37, 77].

In this chapter we choose and fix positive operators H, K on $\mathcal{H}$ with the spectral decompositions

$$H = \int_0^{\|H\|} s\,dE_s \quad \text{and} \quad K = \int_0^{\|K\|} t\,dF_t$$

respectively. We will use both of the notations dE_s, E_Λ (for Borel sets $\Lambda \subseteq [0, \|H\|]$) interchangeably in what follows (and do the same for the other spectral measure F). Let λ (resp. μ) be a finite positive measure on the interval $[0, \|H\|]$ (resp. $[0, \|K\|]$) equivalent (in the absolute continuity sense) to dE_s (resp. dF_t). For instance the measures

$$\lambda(\Lambda) = \sum_{n=1}^{\infty} \frac{1}{n^2}(E_\Lambda e_n, e_n) \quad (\Lambda \subseteq [0, \|H\|]),$$
$$\mu(\Xi) = \sum_{n=1}^{\infty} \frac{1}{n^2}(F_\Xi e_n, e_n) \quad (\Xi \subseteq [0, \|K\|])$$

do the job, where $\{e_n\}_{n=1,2,\cdots}$ is an orthonormal basis for $\mathcal{H}$. We choose and fix a function $\phi(s,t)$ in $L^\infty([0,\|H\|] \times [0, \|K\|]; \lambda \times \mu)$. For each operator

$X \in B(\mathcal{H})$, the algebra of all bounded operators on $\mathcal{H}$, we would like to justify its "double integral" transformation formally written as

$$\Phi(X) = \int_0^{\|H\|}\int_0^{\|K\|} \phi(s,t)\, dE_s X dF_t$$

(see [14]). As long as $X \in \mathcal{C}_2(\mathcal{H})$, the Hilbert-Schmidt class operators, desired justification is quite straight-forward and moreover under such circumstances we have $\Phi(X) \in \mathcal{C}_2(\mathcal{H})$ with the norm bound

$$\|\Phi(X)\|_2 \le \|\phi\|_{L^\infty(\lambda\times\mu)} \times \|X\|_2. \tag{2.1}$$

In fact, with the left multiplication π_ℓ and the right multiplication π_r, $\pi_\ell(E_\Lambda)$ and $\pi_r(F_\Xi)$ (with Borel sets $\Lambda \subseteq [0, \|H\|]$ and $\Xi \subseteq [0, \|K\|]$) are commuting projections acting on the Hilbert space $\mathcal{C}_2(\mathcal{H})$ so that $\pi_\ell(E_\Lambda)\pi_r(F_\Xi)$ is a projection. It is plain to see that one gets a spectral family acting on the Hilbert space $\mathcal{C}_2(\mathcal{H})$ from those "rectangular" projections so that the ordinary functional calculus via $\phi(s,t)$ gives us a bounded linear operator on $\mathcal{C}_2(\mathcal{H})$. With this interpretation we set

$$\Phi(X) = \left(\int_0^{\|H\|}\int_0^{\|K\|} \phi(s,t)\, d(\pi_\ell(E)\pi_r(F))\right) X. \tag{2.2}$$

Note that the Hilbert-Schmidt class operator X in the right side here is regarded as a vector in the Hilbert space $\mathcal{C}_2(\mathcal{H})$, and (2.1) is obvious.

In applications of double integral transformations (for instance to stability problems of perturbation) it is important to be able to specify classes of functions ϕ for which the domain of $\Phi(\cdot)$ can be enlarged to various operator ideals (such as $\mathcal{C}_p$-ideals). In fact, some useful sufficient conditions (in terms of certain Lipschitz conditions on $\phi(\cdot,\cdot)$) were announced in [14] (whose proofs were sketched in [15]), but unfortunately they are not so helpful for our later purpose. More detailed information on double integral transformations will be given in §2.5.

2.1 Schur multipliers and Peller's theorem

We begin with the definition of Schur multipliers (acting on operators on $\mathcal{H}$).

Definition 2.1. When $\Phi\,(= \Phi|_{\mathcal{C}_1(\mathcal{H})}) : X \mapsto \Phi(X)$ gives rise to a bounded transformation on the ideal $\mathcal{C}_1(\mathcal{H})$ ($\subseteq \mathcal{C}_2(\mathcal{H})$) of trace class operators, $\phi(s,t)$ is called a *Schur multiplier* (relative to the pair (H,K)).

When this requirement is met, by the usual duality $B(\mathcal{H}) = (\mathcal{C}_1(\mathcal{H}))^*$ the transpose of Φ gives rise to a bounded transformation on $B(\mathcal{H})$ (i.e., the largest possible domain) as will be explained in the next §2.2. The next important characterization due to V. V. Peller will play a fundamental role in our investigation on means of operators:

Theorem 2.2. (V.V. Peller, [69, 70]) *For* $\phi \in L^{\infty}([0, \|H\|] \times [0, \|K\|]; \lambda \times \mu)$ *the following conditions are all equivalent*:

(i) ϕ *is a Schur multiplier*;
(ii) *whenever a measurable function* $k : [0, \|H\|] \times [0, \|K\|] \to \mathbf{C}$ *is the kernel of a trace class operator* $L^2([0, \|H\|]; \lambda) \to L^2([0, \|K\|]; \mu)$, *so is the product* $\phi(s,t)k(s,t)$;
(iii) *one can find a finite measure space* (Ω, σ) *and functions* $\alpha \in L^{\infty}([0, \|H\|] \times \Omega; \lambda \times \sigma)$, $\beta \in L^{\infty}([0, \|K\|] \times \Omega;\ \mu \times \sigma)$ *such that*

$$\phi(s,t) = \int_{\Omega} \alpha(s,x)\beta(t,x)d\sigma(x) \quad \textit{for all } s \in [0, \|H\|],\ t \in [0, \|K\|]; \tag{2.3}$$

(iv) *one can find a measure space* (Ω, σ) *and measurable functions* α, β *on* $[0, \|H\|] \times \Omega$, $[0, \|K\|] \times \Omega$ *respectively such that the above* (2.3) *holds and*

$$\left\| \int_{\Omega} |\alpha(\cdot, x)|^2 d\sigma(x) \right\|_{L^{\infty}(\lambda)} \left\| \int_{\Omega} |\beta(\cdot, x)|^2 d\sigma(x) \right\|_{L^{\infty}(\mu)} < \infty.$$

A few remarks are in order. (a) The implication (iii) $\Rightarrow$ (iv) is trivial. (b) The finiteness condition in (iv) and the Cauchy-Schwarz inequality guarantee the integrability of the integrand in the right-hand side of (2.3). (c) The condition (iii) is stronger than what was stated in [69, 70], but the proof in [69] (presented below) actually says (ii) $\Rightarrow$ (iii).

Unfortunately Peller's article [69] (with a proof) was not widely circulated. Because of this reason and partly to make the present monograph as much as self-contained, the proof of the theorem is presented in what follows.

Proof of (iv) $\Rightarrow$ (i)
Although this is a relatively easy part in the proof, we present detailed arguments here because its understanding will be indispensable for our later arguments. So let us assume that $\phi(s,t)$ admits an integral representation stated in (iv). For a rank-one operator $X = \xi \otimes \eta^c$ we have $\pi_{\ell}(E_{\Lambda})\pi_r(F_{\Xi})X = (E_{\Lambda}\xi) \otimes (F_{\Xi}\eta)^c$ so that from (2.3) we get

$$\begin{aligned} \Phi(X) &= \int_0^{\|H\|} \int_0^{\|K\|} \int_{\Omega} \alpha(s,x)\beta(t,x)\, (dE_s\xi) \otimes (dF_t\eta)^c \, d\sigma(x) \\ &= \int_{\Omega} \xi(x) \otimes \eta(x)^c \, d\sigma(x) \end{aligned}$$

with

$$\xi(x) = \int_0^{\|H\|} \alpha(s,x)\, dE_s\xi \quad \text{and} \quad \eta(x) = \int_0^{\|K\|} \overline{\beta(t,x)}\, dF_t\eta. \tag{2.4}$$

More precisely, the above integral can be understood for example in the weak sense:

$$(\Phi(X)\xi', \eta') = \int_{\Omega} ((\xi(x) \otimes \eta(x)^c)\xi', \eta')\, d\sigma(x)$$
$$= \int_{\Omega} (\xi', \eta(x))(\xi(x), \eta')\, d\sigma(x). \tag{2.5}$$

The above $\xi(x), \eta(x)$ are vectors for a.e. $x \in \Omega$ as will be seen shortly. We use Theorem A.5 in §A.2 and the Cauchy-Schwarz inequality to get

$$\|\Phi(\xi \otimes \eta^c)\|_1 \leq \int_{\Omega} \|\xi(x) \otimes \eta(x)^c\|_1 d\sigma(x) = \int_{\Omega} \|\xi(x)\| \times \|\eta(x)\|\, d\sigma(x)$$
$$\leq \left(\int_{\Omega} \|\xi(x)\|^2 d\sigma(x)\right)^{1/2} \left(\int_{\Omega} \|\eta(x)\|^2 d\sigma(x)\right)^{1/2}. \tag{2.6}$$

Since $\|\xi(x)\|^2 = \displaystyle\int_0^{\|H\|} |\alpha(s,x)|^2 d(E_s\xi, \xi)$ with the total mass of $d(E_s\xi, \xi)$ being $\|\xi\|^2$, we have

$$\int_{\Omega} \|\xi(x)\|^2 d\sigma(x) = \int_0^{\|H\|} \left(\int_{\Omega} |\alpha(s,x)|^2 d\sigma(x)\right) d(E_s\xi, \xi)$$
$$\leq \left\| \int_{\Omega} |\alpha(\cdot, x)|^2 d\sigma(x) \right\|_{L^\infty(\lambda)} \times \|\xi\|^2 \tag{2.7}$$

by the Fubini-Tonneli theorem. A similar bound for $\int_{\Omega} \|\eta(x)\|^2 d\sigma(x)$ is also available, and consequently from (2.6), (2.7) we get

$$\|\Phi(\xi \otimes \eta^c)\|_1 \leq \|\xi\| \times \|\eta\| \times \left\| \int_{\Omega} |\alpha(\cdot, x)|^2 d\sigma(x) \right\|_{L^\infty(\lambda)}^{1/2}$$
$$\times \left\| \int_{\Omega} |\beta(\cdot, x)|^2 d\sigma(x) \right\|_{L^\infty(\mu)}^{1/2}.$$

Therefore, we have shown

$$\|\Phi(X)\|_1 \leq \left\| \int_{\Omega} |\alpha(\cdot, x)|^2 d\sigma(x) \right\|_{L^\infty(\lambda)}^{1/2}$$
$$\times \left\| \int_{\Omega} |\beta(\cdot, x)|^2 d\sigma(x) \right\|_{L^\infty(\mu)}^{1/2} \times \|X\|_1 \tag{2.8}$$

for rank-one operators X. Note that (2.7) (together with the finiteness requirement in the theorem) shows $\|\xi(x)\| < \infty$, i.e., $\xi(x)$ is indeed a vector for a.e. $x \in \Omega$. Also (2.8) guarantees that $\Phi(X) = \int_{\Omega} \xi(x) \otimes \eta(x)^c d\sigma(x)$ falls into the ideal $\mathcal{C}_1(\mathcal{H})$ of trace class operators.

We claim that the estimate (2.8) remains valid for finite-rank operators. Indeed, thanks to the standard polar decomposition and diagonalization technique, such an operator X admits a representation $X = \sum_{i=1}^n \xi_i \otimes \eta_i^c$ satisfying $\|X\|_1 = \sum_{i=1}^n \|\xi_i\| \times \|\eta_i\|$. Then, we estimate

$$\|\Phi(X)\|_1 \le \sum_{i=1}^{n} \|\Phi(\xi_i \otimes \eta_i^c)\|_1$$
$$\le \sum_{i=1}^{n} \|\xi_i\| \times \|\eta_i\| \times \left\| \int_\Omega |\alpha(\cdot,x)|^2 d\sigma(x)\right\|_{L^\infty(\lambda)}^{1/2}$$
$$\times \left\| \int_\Omega |\beta(\cdot,x)|^2 d\sigma(x)\right\|_{L^\infty(\mu)}^{1/2}$$
(by (2.8) for rank-one operators)
$$= \left\| \int_\Omega |\alpha(\cdot,x)|^2 d\sigma(x)\right\|_{L^\infty(\lambda)}^{1/2} \times \left\| \int_\Omega |\beta(\cdot,x)|^2 d\sigma(x)\right\|_{L^\infty(\mu)}^{1/2} \times \|X\|_1.$$

We now assume $X \in \mathcal{C}_1(\mathcal{H})$. Choose a sequence $\{X_n\}_{n=1,2,\cdots}$ of finite-rank operators converging to X in $\|\cdot\|_1$. Since convergence also takes place in $\|\cdot\|_2$ ($\le \|\cdot\|_1$), we see that $\Phi(X_n)$ tends to $\Phi(X)$ in $\|\cdot\|_2$ (by (2.1)) and consequently in the operator norm $\|\cdot\|$. The lower semi-continuity of $\|\cdot\|_1$ relative to the $\|\cdot\|$-topology thus yields

$$\|\Phi(X)\|_1 \le \liminf_{n\to\infty} \|\Phi(X_n)\|_1$$
$$\le \liminf_{n\to\infty} \Bigg(\left\| \int_\Omega |\alpha(\cdot,x)|^2 d\sigma(x)\right\|_{L^\infty(\lambda)}^{1/2}$$
$$\times \left\| \int_\Omega |\beta(\cdot,x)|^2 d\sigma(x)\right\|_{L^\infty(\mu)}^{1/2} \times \|X_n\|_1 \Bigg)$$
(by (2.8) for finite-rank operators)
$$= \left(\left\| \int_\Omega |\alpha(\cdot,x)|^2 d\sigma(x)\right\|_{L^\infty(\lambda)}^{1/2} \times \left\| \int_\Omega |\beta(\cdot,x)|^2 d\sigma(x)\right\|_{L^\infty(\mu)}^{1/2} \right) \|X\|_1.$$

Therefore, $\Phi(X)$ belongs to $\mathcal{C}_1(\mathcal{H})$, and moreover $\Phi(\cdot)$ restricted to $\mathcal{C}_1(\mathcal{H})$ gives rise to a bounded transformation as desired.

Proof of (i) $\Rightarrow$ (ii)

One can choose a sequence $\{\xi_m\}$ in $\mathcal{H}$ with $\sum_m \|\xi_m\|^2 < \infty$ such that $\{E_\Lambda \xi_m : \Lambda \subseteq [0,\|H\|\}$ $(m = 1,2,\dots)$ are mutually orthogonal and λ is equivalent to the measure $\sum_m (E_\Lambda \xi_m, \xi_m)$. In fact, choose a sequence $\{\xi_m\}$ for which $\sum_m \|\xi_m\|^2 < \infty$ and $\sum_m (E_\Lambda \xi_m, \xi_m)$ is equivalent to λ. We set

$$\Lambda_m = \left\{ s \in [0,\|H\|] : \frac{d(E_s\xi_m, \xi_m)}{d\lambda(s)} > 0 \right\}$$

with the Radon-Nikodym derivative $d(E_s\xi_m,\xi_m)/d\lambda(s)$ with respect to λ. Choose mutually disjoint measurable subsets $\Lambda_m^0 \subseteq \Lambda_m$ $(m = 1,2,\dots)$ with $\bigcup_m \Lambda_m^0 = \bigcup_m \Lambda_m$; then a required sequence is obtained by replacing ξ_m by $E_{\Lambda_m^0}\xi_m$. Furthermore, we easily observe that the condition (ii) (as well as (i)) is unchanged for equivalent measures (by considering the unitary multiplication operator induced by the square root of the relevant Radon-Nikodym deriva-