Introduction to Ordinary Differential Equations

Roger C. McCann

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PREFACE

Differential equations play an enormously important role in engineering, physics, chemistry, the life sciences, and many areas of applied mathematics. Virtually any phenomenon that varies in a continuous or nearly continuous fashion can be modeled using differential equations. It is the goal of this text not only to describe how to solve elementary differential equations, but also to illustrate how differential equations are actually used to model real-world processes.

Traditionally, most of the examples illustrating the use of differential equations involved simple mechanical systems or electrical circuits. I have drawn examples and exercises from numerous books and journals illustrating some of the diverse areas in which differential equations are used. To the best of my knowledge, many of these examples are appearing in an elementary text for the first time. Among the more interesting and unusual of these examples are the rate at which ocean water circulates (Exercise 3, Section 1.3), the concentration of silica in the sediment of the floor of the North Sea (Section 2.4), the stabilization of production in a closed economy (Section 3.5), the relationship between rainfall and runoff in a watershed (Example 3, Section 4.4, and Example 4, Section 4.5), the Lancaster war model (Example 1, Section 5.7), the Ross model for the way malaria affects a community (Example 3, Section 5.10), the water level in a canal that empties into the open sea (Example 2, Section 6.5), and a nonparametric description of a cycloid (Exercise 21, Section 7.1). One-sixth of the 156 examples and one-eighth of the 774 exercises involve applications.

Throughout the text I have tried to keep the explanations and derivations as simple and straightforward as possible. Virtually every concept is illustrated by an example before it is used. Discussions begin with elementary situations and progress to the more complex. For example, the treatment of linear differential equations begins by considering only second-order equations (Chapter 2). This greatly simplifies the concept of linear independence, since only two functions need to be considered. In the following chapter higher-order equations are considered. It is here that the general definition of linear independence is given. This delay in presenting the general definition enables the student to have experience with this essential concept before working in a general setting.

I believe that numerical methods do not constitute a separate area of study, but are merely another technique for finding solutions. For this reason numerical methods appear as sections in Chapters 1 and 5, and not as a separate chapter. I also believe that the numerical methods should be discussed along with other techniques so that the reader realizes that equations that do not have a "nice" form can still be "solved." Of course there is no detailed discussion of numerical methods, but only an indication of the types of methods that are commonly used.

With the exception of Chapter 7 (Boundary-Value Problems) and Sections 1.6, 1.7, and 1.10 (Exact Differential Equations, Integrating Factors, and Existence of Solutions), only a knowledge of calculus of a single variable is assumed. At various points in the text certain elementary properties of determinants and complex numbers are needed; hence, a discussion of these properties is contained in two appendixes.

Chapters 1 and 2 constitute the basis for any introductory course in elementary differential equations. Once these chapters have been covered, the remaining chapters may be studied in any order, with the exception that Chapter 4 should precede Section 5.9. The independence of Chapters 3 through 7 makes the book flexible enough to meet the needs of almost any instructor. My personal view is that Chapter 5 is exceptionally important and should be at least partially covered.

Before writing this text, I did not realize how much of a team effort is required by such a project. I was blessed with exceptionally knowledgeable reviewers who made significant contributions to the final form of the text. I would like to thank them for their assistance: Donald Blevins, Trinity College, Washington, D.C.; John Brothers, Indiana University; Murray Cantor, University of Texas at Austin; J. R. Dorroh, Louisiana State University; Richard Koch, University of Oregon; and Hugh Maynard, University of Texas at San Antonio.

Without a coauthor to share the blame, I alone am responsible for any misprints, weaknesses, or errors that appear in the text. I hope that readers will bring to my attention not only errors and shortcomings of the text, but also its features that they particularly like.

I am grateful to Marilyn Davis, Editor at Harcourt Brace Jovanovich, Publishers, for her guidance and cooperation. I would also like to thank Andrea Haight and Christopher Lang for their excellent editorial work, and Anna Kopczynski for the book's handsome design. Various portions of the manuscript were expertly typed by Linda Brent and Pam Bost. Robert Finley read the manuscript and assisted in the preparation of the answer section. Last, and certainly not least, I would like to thank my wife, Susan, without whose help and understanding this book could never have been written.

Roger C. McCann

CONTENTS

1 First-Order Differential Equations

1.1 Introduction

Differential equations play an enormously important role in engineering, physics, chemistry, and various areas of applied mathematics. Virtually any phenomenon that varies in a continuous or nearly continuous fashion can be modeled using differential equations. Systems as varied as electrical circuits, economic growth, rainfall-runoff in a watershed, and the metabolism of glucose have been modeled using differential equations. It is the goal of this book not only to describe how to solve elementary differential equations, but also to illustrate how differential equations are used to model various processes. In this section we will present some of the basic definitions from the theory of differential equations, while the later sections will be devoted to determining solutions of elementary differential equations and investigating their applications.

We will consider equations such as

$$\frac{d^3y}{dx^3}(x)-2\frac{d^2y}{dx^2}(x)+4\frac{dy}{dx}(x)-3y(x)=0 \tag{1.1}$$

and

$$\frac{d^2z}{dt^2}(t)+4z(t)=0 \tag{1.2}$$

Our goal is to find functions that satisfy the given equation. For example, the function $y(x)=e^x$ satisfies equation (1.1) since

$$\frac{d^3}{dx^3}(e^x)-2\frac{d^2}{dx^2}(e^x)+4\frac{d}{dx}(e^x)-3e^x=e^x-2e^x+4e^x-3e^x=0$$

Likewise, the function $z(t)=\cos 2t$ satisfies equation (1.2) since

$$\frac{d^2}{dt^2}(\cos 2t)+4\cos 2t=-4\cos 2t+4\cos 2t=0$$

Equations in which the unknown is a function of a real variable and which contain not only the functions themselves, but also certain of their derivatives, are called **ordinary differential equations**. For example, in equation (1.1), x is the real variable and y is the unknown function. In equation (1.2), t is the real variable and z is the unknown function. Other examples of ordinary differential equations are

$$\frac{dy}{dx}(x)+x^3y(x)=x^2$$

$$\left(\frac{d^3w}{dz^3}(z)\right)^2+z\frac{dw}{dz}(x)+4w(z)=\sin z \qquad \textbf{(1.3)}$$

$$\frac{d^2\theta}{dt^2}(t)+\sin\theta(t)=0$$

Relations that do not involve derivatives, such as

$$x^2+y^2=x+y$$

or relations that contain partial derivatives, such as

$$\frac{\partial^2 y}{\partial x^2}(x,t)=\frac{\partial^2 y}{\partial t^2}(x,t)$$

do not define ordinary differential equations.

For brevity, an ordinary differential equation will be called a differential equation. In order to simplify the notation,

$$y, \quad \frac{dy}{dx}, \quad \frac{d^2y}{dx^2},\ldots, \quad \frac{d^ny}{dx^n}$$

will frequently be used in place of

$$y(x), \quad \frac{dy}{dx}(x), \quad \frac{d^2y}{dx^2}(x),\ldots, \quad \frac{d^ny}{dx^n}(x)$$

respectively, when writing differential equations. With this understanding, the differential equations in (1.1), (1.2), and (1.3) can be rewritten as

$$\frac{d^3y}{dx^3}-2\frac{d^2y}{dx^2}+4\frac{dy}{dx}-3y=0$$

$$\frac{d^2z}{dt^2}+4z=0$$

$$\frac{dy}{dx}+x^3y=x^2$$

$$\left(\frac{d^3w}{dz^3}\right)^2+z\frac{dw}{dz}+4w=\sin z$$

$$\frac{d^2\theta}{dt^2}+\sin\theta=0$$

The **order** of a differential equation is the order of the highest derivative appearing in the equation. Thus the differential equations in (1.1), (1.2), and (1.3) are of third, second, first, third, and second orders, respectively. Throughout this book we will always assume that an nth order differential equation can be written in the form

$$\frac{d^ny}{dx^n}=f\left(x, y, \frac{dy}{dx},\dots,\frac{d^{n-1}y}{dx^{n-1}}\right) \tag{1.4}$$

Equations (1.1) and (1.2) are of this type since they can be rewritten as

$$\frac{d^3y}{dx^3}=2\frac{d^2y}{dx^2}-4\frac{dy}{dx}+3y$$

and

$$\frac{d^2z}{dt^2}=-4z$$

The second equation in (1.3) is a third-order differential equation that can not be written in the form in (1.4).

A **solution** on an open interval I of the differential equation in (1.4) is a function u defined on I such that

$$\frac{d^nu}{dx^n}(x)=f\left(x, u(x), \frac{du}{dx}(x),\dots,\frac{d^{n-1}}{dx^{n-1}}u(x)\right)$$

for every x in the interval I.

▶ **Example 1** The function $u(x)=e^x$ is a solution on $(-\infty,\infty)$ of $dy/dx=y$ since $du/dx=e^x=u(x)$ for every x in $(-\infty,\infty)$. ◀

▶ **Example 2** The function $u(x)=x^{-1}$ is a solution on $(0,\infty)$ of $dy/dx=-y^2$ since $du/dx=-x^{-2}=-[u(x)]^2$ for every x in $(0,\infty)$. ◀

A relation $g(x,u)=0$ is called an **implicit solution** on an open interval I of equation (1.4) if it determines at least one real function on I and this function is a solution of the differential equation.

▶ **Example 3** The relation $x^2+u^2-1=0$ is an implicit solution of $dy/dx=-x/y$ on $(-1,1)$ since it determines the function $u_1(x)=\sqrt{1-x^2}$ on $(-1,1)$ and

$$\frac{du_1}{dx}=\frac{-x}{\sqrt{1-x^2}}=-\frac{x}{u_1(x)}$$

for every x in $(-1,1)$. The relation $x^2+u^2-1=0$ also determines the function $u_2(x)=-\sqrt{1-x^2}$ on $(-1,1)$. A calculation similar to that done for u_1 shows that the function u_2 is also a solution of the differential equation $dy/dx=-x/y$ on $(-1,1)$. ◀

▶ **Example 4** One of the most important problems in mechanics is that of determining the motion of an object constrained to move along a straight line and acted upon by a force F. If the mass of the object is m and the acceleration and position of the mass at time t are $a(t)$ and $y(t)$, respectively, then by Newton's second law of motion we have

$$F=ma$$

or, since $a(t)=d^2y/dt^2$,

$$\frac{d^2y}{dt^2}=\frac{F}{m} \tag{1.5}$$

The force acting on the object may vary with time, with the position of the object, or with the velocity of the object. For example, the force due to gravity is a function of the distance from the center of the earth, and the force acting on the moving object due to air resistance is a function of the velocity of the object. Thus the equation of motion in (1.5) has the form of equation (1.4) for the case $n=2$.

The simplest form of equation (1.5) occurs when the force F is constant. In such a case the equation can be solved by integrating twice. Doing this gives

$$y(t)=\frac{F}{2m}t^2+c_1t+c_2 \tag{1.6}$$

where c_1 and c_2 are constants from the two integrations. If we wish to determine the position $y(t)$ at time t of the object, we must have some additional information in order to evaluate the constants c_1 and c_2. Such information may be given in terms of the initial position, $y(0)$, and the initial velocity, $dy(0)/dt$. If we require that $y(0)=y_0$ and $dy(0)/dt=y_1$, then using equation (1.6) we find that

$$y_0=y(0)=c_2$$

$$y_1=\frac{dy(0)}{dt}=c_1$$

Thus with these initial values we are able to uniquely determine the position $y(t)$ at time t of the object:

$$y(t)=\frac{F}{2m}t^2+y_1t+y_0$$

Notice that it took two pieces of initial data to determine a unique solution of equation (1.5). ◀

This example illustrates two fundamental properties of differential equations. First, a differential equation usually has infinitely many solutions. Second, if values of the solution and certain of its derivatives are preassigned, then exactly one solution may be determined. One set of such values which usually determines a unique solution of the differential equation in (1.4) is

$$y(a)=y_0,\qquad \frac{dy}{dx}(a)=y_1,\quad \ldots,\quad \frac{d^{n-1}y}{dx^{n-1}}(a)=y_{n-1} \tag{1.7}$$

where $y_0, y_1, \ldots, y_{n-1}$ are constants. Such conditions are often called **initial conditions** because x frequently measures time and a is often taken as the starting time of the process involved. Notice that in Example 4 the initial conditions determined uniquely the position of the moving object. The problem of finding a solution of equation (1.4) that satisfies the initial conditions in (1.7) is called the **initial-value problem** for equation (1.4).

Common problems involving differential equations are: (1) to find a solution that satisfies preassigned conditions, such as initial conditions; (2) to find all of the solutions; (3) to determine properties of solutions without actually computing the solutions; and (4) to approximate a solution numerically. All of these problems will be discussed in varying detail throughout the remainder of the text.

In general, only differential equations that have special forms can be solved explicitly. We will now define one special form of a differential equation that arises frequently in applications and can often be solved explicitly.

Let $a_{n-1}, \ldots, a_1, a_0$, and r be functions defined on an open interval I. If equation (1.4) can be rewritten as

$$\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = r(x) \tag{1.8}$$

then it is called a **linear differential equation.** If the differential equation can not be written in this form, it is called a **nonlinear differential equation.** Equations (1.1), (1.2), and the first equation in (1.3) are linear differential equations. The remaining two equations in (1.3) are nonlinear differential equations. The differential equation in Example 1 is linear, while those in Examples 2 and 3 are nonlinear.

If the coefficient functions $a_0, a_1, \ldots, a_{n-1}$ are constant functions, then, as we will see later in the text, there are techniques that will determine all the solutions of the linear differential equation in (1.8).

Exercises SECTION 1.1

In Exercises 1–8 determine the order of the differential equation and whether it is linear.

1. $\left(\frac{dy}{dx}\right)^2 + y = x^3$

2. $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 + x^5 y = 0$

3. $\frac{d^5y}{dx^5} + \frac{d^3y}{dx^3} + y + x^7 = 0$

4. $x^2\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^3 + y = e^x$

5. $\sin\left(\frac{d^3y}{dx^3}\right) + y = 0$

6. $\dfrac{d^3y}{dx^3}+x\dfrac{d^2y}{dx^2}+e^y=x^3$

7. $\dfrac{dy}{dx}+y^2+y=x+e^x$

8. $\dfrac{d^2y}{dx^2}+(\cos x)\dfrac{dy}{dx}+y=3x$

In Exercises 9–16 verify that the given function is a solution of the differential equation.

9. $\dfrac{d^2y}{dx^2}+y=0,\quad u(x)=\cos x$ on $(-\infty,\infty)$

10. $\dfrac{d^2y}{dx^2}+4y=0,\quad u(x)=\sin 2x$ on $(-\infty,\infty)$

11. $\dfrac{dy}{dx}+2y^{3/2}=0,\quad u(x)=x^{-2}$ on $(0,\infty)$

12. $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+2y=0,\quad u(x)=e^{-x}+e^{-2x}$ on $(-\infty,\infty)$

13. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=0,\quad u(x)=2e^x+3xe^x$ on $(-\infty,\infty)$

14. $\dfrac{d^3y}{dx^3}-2\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}+2y=0,\quad u(x)=3e^x+2e^{-x}+e^{2x}$ on $(-\infty,\infty)$

15. $\dfrac{d^3y}{dx^3}+3\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+y=0,\quad u(x)=e^{-x}(3+5x+4x^2)$ on $(-\infty,\infty)$

16. $\dfrac{d^4y}{dx^4}+4\dfrac{d^3y}{dx^3}+6\dfrac{d^2y}{dx^2}+4\dfrac{dy}{dx}+y=0,\quad u(x)=x^3e^{-x}$ on $(-\infty,\infty)$

In Exercises 17–20 find numbers r such that $u(x)=e^{rx}$ is a solution of the given equation.

17. $\dfrac{d^2y}{dx^2}+5\dfrac{dy}{dx}+4y=0$

18. $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}-4y=0$

19. $\dfrac{d^3y}{dx^3}-2\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}+2y=0$

20. $\dfrac{d^4y}{dx^3}-5\dfrac{d^2y}{dx^2}+4y=0$

In Exercises 21–24 find numbers r such that $u(x)=x^r$ is a solution of the given differential equation.

21. $x^2\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}-6y=0$

22. $2x^2\dfrac{d^2y}{dx^2}+5x\dfrac{dy}{dx}+y=0$

23. $x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}-2y=0$

24. $x^3\dfrac{d^3y}{dx^2}+x\dfrac{dy}{dx}-y=0$

In Exercises 25–28 determine the constants c_1 and c_2 such that $u(x)=c_1e^x+c_2e^{-x}$ is a solution of $d^2y/dx^2-y=0$ that satisfies the initial conditions.

25. $y(0)=0,\quad \dfrac{dy}{dx}(0)=1$

26. $y(0)=1,\quad \dfrac{dy}{dx}(0)=0$

27. $y(0)=0,\quad \dfrac{dy}{dx}(0)=0$

28. $y(0)=2,\quad \dfrac{dy}{dx}(0)=4$

In Exercises 29–32 find the solution of each initial-value problem.

29. $\dfrac{dy}{dx}=x^2,\quad y(0)=2$

30. $\dfrac{dy}{dx}=x^3+1,\quad y(0)=-1$

31. $\dfrac{dy}{dx}=\sin x,\quad y(\pi)=0$

32. $\dfrac{d^2y}{dx^2}=\cos x,\quad y(0)=1,\quad \dfrac{dy(0)}{dx}=1$

1.2 First-Order Linear Differential Equations

In the following section we will see that various physical, chemical, biological, and economic processes can be modeled by the first-order linear differential equation (where $'$ denotes d/dx)

$$y'+p(x)y=q(x) \tag{1.9}$$