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A Survey

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INTRODUCTION

The aim of these notes is to give a survey of the main developments in the theory of "ordinary", ie. characteristic 0 representations of finite Chevalley groups which have occurred in recent years. In the year 1969-70 a seminar on finite groups arising from algebraic groups was held at the Institute for Advanced study. In this seminar T.A. Springer gave some lectures on Harish-Chandra's philosophy of cusp forms, which had at that time been applied also to these finite groups. Springer then stated the so called "Macdonald conjectures" which predicted that there would be families of representations of these groups parametrized by the characters of the various "maximal tori". An important breakthrough came in 1976 when Lusztig and Deligne in their famous paper [18] published a proof of these conjectures, by constructing virtual representations of the groups on the ℓ-adic cohomology of certain varieties.

An outline of the contents is as follows. Chapter I is a review of the main results that we need on the "absolute theory" of reductive algebraic groups over an algebraically closed field. In Chapter II and in the rest of the notes we consider the situation where G is a connected reductive group defined over $\mathbb{F}_q$, and where F is a Frobenius endomorphism of G. The group G^F of fixed points under F is the finite group whose representation theory will be studied. The classification of the maximal tori of G^F is described, leading us to the problem of constructing a family of virtual representations of

G^F corresponding to each torus. In Chapter III this is done in the easiest case, ie. in the case of the "split torus", leading to the principal series representations. In Chapter IV the theory of Harish-Chandra is described and this brings out the importance of constructing cuspidal (discrete series) representations.

Chapter V is perhaps the raison d'etre for these notes. In recent years I have detected a growing dissatisfaction among finite group theorists about assuming the existence and main properties of ℓ-adic cohomology on an axiomatic basis. I have therefore endeavored in this chapter, starting from a brief review of the classical theory of sheaves on a topological space and of sheaf cohomology, to give an idea of how ℓ-adic cohomology groups are constructed and to give a feeling for their properties by pointing out classical analogues when possible. It is hoped that this chapter will be of independent interest.

Chapter VI contains the main results in the paper of Lusztig and Deligne. If T is an F-stable maximal torus of G, a virtual representation $R_T^G(\theta)$ of G^F is constructed corresponding to each character θ of T^F (ie. homomorphism of T^F into $\bar{\mathbb{Q}}_\ell^*$, where ℓ is different from p, the characteristic of $\mathbb{F}_q$). If θ is regular, ie. not fixed by any non-trivial element of $N(T)^F/T^F$, then $R_T^G(\theta)$ is irreducible, up to sign.

Orthogonality properties of the $R_T^G(\theta)$ are established, and their dimensions are computed. The connection between this theory and the Harish-Chandra theory is established; in particular if T is a "minisotropic torus" and θ is a regular character of T^F then $\pm R_T^G(\theta)$ is cuspidal (but not all cuspidal representations arise this way). The **proof** of an important result in the paper [18] ((6.3) of these notes) which leads to a reduction formula for the character values of the $R_T^G(\theta)$ has so far been inaccessible to many group theorists because of the technical machinery involved. I have described the main ideas in this proof, omitting some of the details, and to my mind this is the most interesting feature of this chapter. The rest of the material follows either the Lusztig-Deligne paper [18] or the monograph of Lusztig [48].

The determination of the explicit values of the characters of the $R_T^G(\theta)$ remains one of the main unsolved problems in the theory. The work of Springer and Kazhdan which enables us to write down the values at unipotent elements in terms of "trigonometric sums" on the Lie algebra (provided p and q are large) is described in Chapter VII. Finally in Chapter VIII I have tried to bring the material up-to-date by describing recent work of Lusztig on the classification of representations of classical groups and of "unipotent" representations for all types. This chapter also contains a section on Hecke algebras ie. centralizer algebras of the representations of G^F induced from certain representations of parabolic subgroups. These algebras arise naturally when we try to decompose these induced representations.

The notes are intended to be accessible to advanced graduate students. A knowledge of the representation theory of finite groups to the extent of say, Parts I and II of the book by Serre [66] is assumed; some knowledge of algebraic groups is desirable but not absolutely necessary. In Chapters II through VII I have given proofs of most of the results; Chapter VIII is essentially a review of recent results but I have included some discussions of proofs. The bibliography includes mainly the papers that I have quoted in the notes. For supplementary references the reader can consult a survey article by Curtis [14].

Acknowledgments

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I owe a great debt to George Lusztig, who has generously shared with me his time and his ideas during the last few years. His beautiful papers have led me into new worlds whose existence I was only dimly aware of earlier. I was helped in overcoming my initial trepidation at entering these worlds by conversations with, and encouragement from, David Kazhdan.

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Table of Contents

CHAPTER I. REVIEW OF RESULTS ON ALGEBRAIC GROUPS.

The references for this chapter are [5], [32], and [40]. We will define affine varieties, introduce linear algebraic groups over an algebraically closed field, and review some of their basic properties.

We will mostly follow the notation of [40]. Let K be an algebraically closed field. An <u>affine variety</u> $X \subset K^n$ is the set of zeros in K^n of a (finite) set of polynomials in $K[x_1,x_2,\ldots,x_n]$. The Zariski topology on K^n is defined by stipulating that the closed sets are affine varieties. Then we have the induced Zariski topology on any subset X of K^n.

An affine variety X is said to be <u>irreducible</u> if X is not the union of two proper non-empty closed subsets. Any affine variety can be written as the union of a finite number of closed irreducible subsets called its components. If the affine variety X is defined by a set of polynomials which generate an ideal I of $K[x_1,x_2,\ldots,x_n]$, X is irreducible if and only if I is a prime ideal. The ring $K[X] = K[x_1,x_2,\ldots,x_n]/I$ is called the <u>affine coordinate ring</u> of X. Each element $f + I \in K[X]$ (where $f \in K[x_1,x_2,\ldots,x_n]$) gives rise to a K-valued function on X; if $(a_1,a_2,\ldots,a_n) \in X$, we map it on $f(a_1,a_2,\ldots,a_n) \in K$. So $K[X]$ is also called the algebra of polynomial functions on X. It is a finitely generated K-algebra, and X can be identified with $\mathrm{Hom}_K(K[X],K)$. If X is irreducible, the <u>dimension</u> of X is the transcendence degree of the quotient field $K(X)$ of $K[X]$, over K.

Let $X \subset K^n$, $Y \subset K^m$ be affine varieties. We say a map $\phi: X \to Y$ is a morphism if $\phi(a_1, a_2, \ldots, a_n) = (\psi_1(a_1, a_2, \ldots, a_n) \ldots, \psi_m(a_1, a_2, \ldots, a_n))$ where $\psi_1, \ldots, \psi_m \in K[x_1, x_2, \ldots, x_n]$. The morphism ϕ induces a comorphism $\phi^*: K[Y] \to K[X]$ by $\phi^*(f) = \phi \cdot f$ and ϕ^* is a K-algebra homomorphism. If X,Y are affine varities, we can define a product variety $X \times Y$ and $K[X \times Y] \cong K[X] \otimes K[Y]$.

We can define the Zariski topology on projective n-space $\mathbb{P}^n$ over K by taking closed sets to be the common zeros of a set of homogeneous polynomials over K. The closed subsets of $\mathbb{P}^n$ are called projective varieties. A quasi-projective variety is an open set in a projective variety.

We regard GL(n,K), the group of all $n \times n$ invertible matrices over K, as being embedded in K^{n^2}. Then a group $G \subset GL(n,K)$ is called a linear algebraic group if it is the intersection with GL(n,K) of a closed subset of K^{n^2}. A map $G \to H$ of linear algebraic groups is a morphism if it is a morphism in the sense of affine varieties and a group homomorphism.

Examples of linear algebraic groups.

1. GL(n,K).
2. SL(n,K), the group of $n \times n$ matrices with determinant 1 over K.
3. $Sp(2n,K) = \{A \in GL(2n,K) \mid {}^tA \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} A = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}\}$, where

   where J is the $n \times n$ matrix $\begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$.

4. $O(2n+1,K) = \{A \in GL(2n+1,K) \mid {}^tA \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & J \\ 0 & J & 0 \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & J \\ 0 & J & 0 \end{pmatrix}\}$

   where char $K \neq 2$.

   $SO(2n+1,K) = O(2n+1,K) \cap SL(2n+1,K)$.

5. $O(2n,K) = \{A \in GL(2n,K) \mid {}^tA \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} A = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}\}$

   where char $K \neq 2$.

   $SO(2n,K) = O(2n,K) \cap SL(2n,K)$.

Remark. The groups of 4 and 5 have to be defined differently in characteristic 2. See [9], p. 8.

6. The group of diagonal matrices over K.
7. The group of upper triangular matrices over K.
8. The group of upper triangular matrices over K with entries 1 along the diagonal.
9. $G_2(K)$: A group of type G_2 over $\mathbb{C}$ can be defined as the group of automorphisms of a Cayley algebra. In analogy with this, we can define a group $G_2(K)$ over K to be generated by 7×7 matrices over K satisfying certain conditions (see e.g. [58], p. 400).

Let G be a linear algebraic group. Then there is a unique irreducible component G^o of G containing the identity element, and G^o is a normal subgroup of finite index. G is connected (in the Zariski topology) if and only if G is irreducible, and if and only if G has no closed subgroup of finite index. (See [40], 7.3.)

Example. $O(2n,K)$ is not connected, whereas $SL(2n,K)$ is connected.

Definitions. 1. G is simple if it has no closed connected non-trivial normal subgroups.

Example. $SL(n,K)$.

2. G is semisimple if it has no closed connected non-trivial normal abelian subgroups.

Examples. $SL(n,K)$, $Sp(2n,K)$, $SO(2n+1,K)$, $SO(2n,K)$.

3. G is unipotent if it is isomorphic to a closed subgroup of the group of upper triangular $n \times n$ matrices over K with entries 1 along the diagonal, for some n.

Example. The additive group of K is isomorphic to the group $\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \}$, $a \in K$.

4. G is reductive if it has no closed connected non-trivial normal unipotent subgroups.

Examples. $GL(n,K)$, $CSp(2n,K)$ (the group of symplectic similitudes).

5. G is a torus if it is isomorphic to a product of multiplicative groups of K.

Example. The group of $n \times n$ diagonal matrices over K.

In the famous Chevalley Seminar of 1956-58 [11] the semisimple groups over K were classified up to isomorphism. The simple ones fall into families of types A,B,C,D (classical groups) or G_2, F_4, E_6, E_7, E_8 (exceptional groups). In the notation we have used above, $SL(n,K) = A_{n-1}(K)$, $SO(2n+1,K)$

$= B_n(K)$, $SO(2n,K) = D_n(K)$, $Sp(2n,K) = C_n(K)$. (See [40], §32.)

For the rest of this chapter we assume that G is a connected reductive linear algebraic group over K and state certain properties of G.

Definition. A maximal connected solvable subgroup of G is called a Borel subgroup of G.

Proposition 1.1. (See [40], 21.3, 23.1) (i) All the maximal tori in G are conjugate in G.

(ii) All the Borel subgroups of G are conjugate.

(iii) If B is a Borel subgroup then $N_G(B) = B$.

The dimension of a maximal torus is called the rank of G. G has a maximal connected normal solvable subgroup called its radical, and the quotient is semisimple. The rank of the quotient group is called the semisimple rank of G. Let T be a maximal torus of G. Then $N(T)/T$ is a finite group called the Weyl group of T and denoted by $W(T)$. It is a finite reflection group. We have the Bruhat decomposition $G = \bigcup_{w \in W(T)} B\dot{w}B$ where $\dot{w}$ is a representative for $w \in W$ in $N(T)$, and the double cosets are disjoint. We have $N(T) \cap B = T$. (See [40], 28.3.)

A subgroup P of G is called a parabolic subgroup if it contains a Borel subgroup. A parabolic subgroup P has a Levi decomposition, ie. a semidirect product decomposition $P = LV$ where V is the maximal connected unipotent normal subgroup of P and is called the unipotent radical of P. L is reductive and is called a Levi subgroup. It is not unique, but any two Levi subgroups are conjugate. In particular, if B is a Borel subgroup we have $B = TU$ where T is a maximal torus and U is a maximal unipotent subgroup of G. (See [40], 30.2.).

For any maximal torus T let $X(T) = \mathrm{Hom}(T,K^*)$, the group of morphisms of T into K^* (ie. characters of T) and let $V = X(T) \underset{\mathbb{Z}}{\times} \mathbb{R}$. Then $W(T)$ acts on V and this is the natural representation of $W(T)$ as a reflection group. There is a subset Φ of $X(T)$ which is an <u>abstract root system</u> ([40], p. 229) in V, except that Φ may not generate V if G is not semisimple. The elements of Φ are called the <u>roots</u> of G with respect to T. We can choose a set of <u>simple roots</u> Δ in Φ such that every root is either a positive root, ie. a linear combination of roots in Δ with positive coefficients, or a negative root, ie. a linear combination of roots in Δ with negative coefficients. Then $\Phi = \Phi^+ \cup \Phi^-$ where $\Phi^+(\Phi^-)$ is the set of positive (negative) roots. For each $\alpha \in \Phi$ there is a T-stable connected unipotent subgroup U_α of G and isomorphisms $x_\alpha : K \to U_\alpha$ such that $tx_\alpha(u)t^{-1} = x_\alpha(\alpha(t)u)$ $(t \in T, u \in K)$. G is generated by the <u>root subgroups</u> U_α $(\alpha \in \Phi)$ and T. The subgroup U generated by the U_α $(\alpha \in \Phi^+)$ is a maximal unipotent subgroup. Then TU is a Borel subgroup, and in fact the choice of a Borel subgroup containing T is equivalent to the choice of a set of simple roots in Φ. The subgroup of G generated by the U_α $(\alpha \in \Phi^-)$ is called the opposite of U and is denoted by U^-. For each $w \in W(T)$ Let $U_w^- = U \cap \dot{w}U^-\dot{w}^{-1}$. Then we have the following refinement of the Bruhat decomposition. Every element x in G can be written uniquely as $x = u\dot{w}tu'$ where $u \in U$, $u' \in U_w^-$, $t \in T$. (See [40], 26.3, 27.3, 28.4.)

Remark. For an easy treatment of the Bruhat decomposition in the case of SL_n and Sp_{2n} see [80], p. 73.

Finally we mention the Jordan decomposition of elements of G. Suppose $x \in GL(V)$ where V is a finite-dimensional vector space over K. Then we say x is semisimple if x is a diagonalizable automorphism of V and that x is unipotent if all of its eigenvalues are 1. If x is an arbitrary element of $GL(V)$ we have $x = x_s x_u = x_u x_s$ where x_s is semisimple and x_u is unipotent; x_s and x_u are determined uniquely by these conditions. Now let $x \in G$. Then we have $x = su = us$ where, in any embedding $G \hookrightarrow GL(V)$, s maps onto a semisimple element and u onto a unipotent element. We call s the semisimple part of x and u the unipotent part of x. (See [40], 15.3.) A torus of G consists entirely of semisimple elements and a unipotent subgroup consists entirely of unipotent elements.

CHAPTER II. CLASSIFICATION OF TORI.

In this chapter we will assume that $K = \bar{\mathbb{F}}_p$ for some prime p.

Let X be an affine variety over K. We say X is defined over $\mathbb{F}_q \subset K$, or has an $\mathbb{F}_q$-rational structure, if X can be defined by a set of polynomials with coefficients in $\mathbb{F}_q$. If X and Y are affine varieties defined over $\mathbb{F}_q$ we can talk of a morphism $\phi: X \to Y$ being defined over $\mathbb{F}_q$; we require a set of polynomials $\psi_1, \psi_2, \ldots, \psi_m$ defining ϕ (see the definition in Chapter I) to be polynomials over $\mathbb{F}_q$.

Suppose X is defined over $\mathbb{F}_q$. Then the affine coordinate ring $A = K[X]$ has a subring A_0 which is an $\mathbb{F}_q$-algebra such that $A_0 \otimes_{\mathbb{F}_q} K = A$. We have a map $F: A \to A$ called the <u>geometric</u> Frobenius morphism and defined by $F(a_0 \otimes \lambda) = a_0^q \otimes \lambda$, and a map $\phi: A \to A$ called the <u>arithmetic</u> Frobenius morphism defined by $\phi(a_0 \otimes \lambda) = a_0 \otimes \lambda^q$ $(a_0 \in A_0, \lambda \in K)$. Then $F\phi = \phi F$ is the map $a \to a^q$ of A. F is an algebra homomorphism and is a bijection from A onto A^q, and for each $a \in A$ there is an $n \geq 1$ such that $F^n a = a^{q^n}$. (For the details of this see the proof of 2.10.) The map ϕ is a bijection and is a semilinear ring homomorphism (ie. $\phi(\lambda a) = \lambda^q \phi(a)$ for $\lambda \in K$, $a \in A$), and for each $a \in A$ there is an $n \geq 1$ such that $\phi^n(a) = a$. Giving an $\mathbb{F}_q$-rational structure on X is equivalent to defining an $\mathbb{F}_q$-algebra $A_0 \subset A$ with $A_0 \otimes K = A$, and this in turn is equivalent to giving either a map F or ϕ with the above properties. For instance, if