Mathematical Analysis

a straightforward approach

K.G. BINMORE

MATHEMATICAL ANALYSIS

A STRAIGHTFORWARD APPROACH

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PREFACE

This book is intended as an easy and unfussy introduction to mathematical analysis. Little formal reliance is made on the reader's previous mathematical background, but those with no training at all in the elementary techniques of calculus would do better to turn to some other book.

An effort has been made to lay bare the bones of the theory by eliminating as much unnecessary detail as is feasible. To achieve this end and to ensure that all results can be readily illustrated with concrete examples, the book deals only with 'bread and butter' analysis on the real line, the temptation to discuss generalisations in more abstract spaces having been reluctantly suppressed. However, the need to prepare the way for these generalisations has been kept well in mind.

It is vital to adopt a systematic approach when studying mathematical analysis. In particular, one should always be aware at any stage of what may be assumed and what has to be proved. Otherwise confusion is inevitable. For this reason, the early chapters go rather slowly and contain a considerable amount of material with which many readers may already be familiar. To neglect these chapters would, however, be unwise.

The exercises should be regarded as an integral part of the book. There is a great deal more to be learned from attempting the exercises than can be obtained from a passive reading of the text. This is particularly the case when, as may frequently happen, the attempt to solve a problem is unsuccessful and it is necessary to turn to the solutions provided at the end of the book.

To help those with insufficient time at their disposal to attempt all the exercises, the less vital exercises have been marked with the symbol †. (The same notation has been used to mark one or two passages in the text which can be omitted without great loss at a first reading.) The symbol * has been used to mark exercises which are more demanding than most but which are well worth attempting.

The final few chapters contain very little theory compared with the number of exercises set. These exercises are intended to illustrate the power of the techniques introduced earlier in the book and to provide the opportunity of some revision of these ideas.

This book arises from a course of lectures in analysis which is given at the London School of Economics. The students who attend this course are mostly not specialist mathematicians and there is little uniformity in their previous mathematical training. They are, however, quite well-motivated. The course is a 'one unit' course of approximately forty lectures supplemented by twenty informal problem classes. I have found it possible to cover the material of this book in some thirty lectures. Time is then left for some discussion of point set topology in simple spaces. The content of the book provides an ample source of examples for this purpose while the more general theorems serve as reinforcement for the theorems of the text.

Other teachers may prefer to go through the material of the book at a more leisurely pace or else to move on to a different topic. An obvious candidate for further discussion is the algebraic foundation of the real number system and the proof of the Continuum Property. Other alternatives are partial differentiation, the complex number system or even Lebesgue measure on the line.

I would like to express my gratitude to Elizabeth Boardman and Richard Holmes for reading the text for me so carefully. My thanks are also due to 'Buffy' Fennelly for her patience and accuracy in preparing the typescript. Finally, I would like to mention M.C. Austin and H. Kestelman from whom I learned so much of what I know.

July 1976 K.G.B.

CONTENTS

1 REAL NUMBERS

1.1 Set notation

A *set* is a collection of objects which are called its *elements*. If x is an element of the set S, we say that x *belongs* to S and write

$$x \in S.$$

If y does not belong to S, we write $y \notin S$.

The simplest way of specifying a set is by listing its elements. We use the notation

$$A = \{\tfrac{1}{2}, 1, \sqrt{2}, e, \pi\}$$

to denote the set whose elements are the real numbers $\frac{1}{2}, 1, \sqrt{2}, e$ and π. Similarly

$$B = \{\text{Romeo, Juliet}\}$$

denotes the set whose elements are Romeo and Juliet.

This notation is, of course, no use in specifying a set which has an infinite number of elements. Such sets may be specified by naming the property which distinguishes elements of the set from objects which are not in the set. For example, the notation

$$C = \{x : x > 0\}$$

(which should be read 'the set of all x such that $x > 0$') denotes the set of all positive real numbers. Similarly

$$D = \{y : y \text{ loves Romeo}\}$$

denotes the set of all people who love Romeo.

It is convenient to have a notation for the *empty* set $\emptyset$. This is the set which has *no* elements. For example, if x denotes a variable which ranges over the set of all real numbers, then

$$\{x : x^2 + 1 = 0\} = \emptyset.$$

This is because there are no real numbers x such that $x^2 = -1$.

If S and T are two sets, we say that S is a *subset* of T and write

$$S \subset T$$

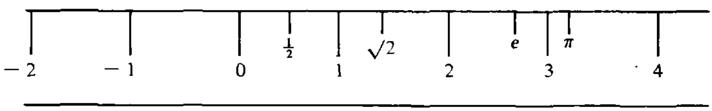
if every element of S is also an element of T.

As an example, consider the sets $P = \{1, 2, 3, 4\}$ and $Q = \{2, 4\}$. Then $Q \subset P$. Note that this is *not* the same thing as writing $Q \in P$, which means that Q is an element of P. The elements of P are simply 1, 2, 3 and 4. But Q is not one of these.

The sets A, B, C and D given above also provide some examples. We have $A \subset C$ and (presumably) $B \subset D$.

1.2 The set of real numbers

It will be adequate for these notes to think of the real numbers as being points along a straight line which extends indefinitely in both directions. The line may then be regarded as an ideal ruler with which we may measure the lengths of line segments in Euclidean geometry.



The set of all real numbers will be denoted by $\mathbb{R}$. The table below distinguishes three important subsets of $\mathbb{R}$.

Subset	Notation	Elements
Natural numbers (or whole numbers)	$\mathbb{N}$	$1, 2, 3, 4, 5, \ldots$
Integers	$\mathbb{Z}$	$\ldots -2, -1, 0, 1, 2, 3, \ldots$
Rational numbers (or fractions)	$\mathbb{Q}$	$0, 1, 2, -1, \frac{1}{2}, \frac{3}{4}, \frac{5}{3}, -\frac{1}{2}, -\frac{3}{7}, \ldots$

Not all real numbers are rational. Some examples of irrational numbers are $\sqrt{2}$, e and π.

While we do not go back to first principles in these notes, the treatment will be rigorous in so far as it goes. It is therefore important to be clear, at every stage, about what our assumptions are. We shall then know what has to be proved and what may be taken for granted. Our most vital assumptions are concerned with the properties of the real number system. The rest of this chapter and the following two chapters are consequently devoted to a description of the properties of the real number system which we propose to assume and to some of their immediate consequences.

1.3 Arithmetic

The first assumption is that the real numbers satisfy all the usual laws of addition, subtraction, multiplication and division.

The rules of arithmetic, of course, include the proviso that division by zero is not allowed. Thus, for example, the expression

$$\frac{2}{0}$$

makes no sense at all. In particular, it is *not* true that

$$\frac{2}{0} = \infty.$$

We shall have a great deal of use for the symbol ∞, but it must clearly be understood that ∞ does *not* represent a real number. Nor can it be treated as such except in very special circumstances.

1.4 Inequalities

The next assumptions concern inequalities between real numbers and their manipulation.

We assume that, given any two real numbers a and b, there are three mutually exclusive possibilities:

(i) $a > b$ (a is greater than b)

(ii) $a = b$ (a equals b)

(iii) $a < b$ (a is less than b).

Observe that $a < b$ means the same thing as $b > a$. We have, for example, the following inequalities.

$$1 > 0;\ 3 > 2;\ 2 < 3;\ -1 < 0;\ -3 < -2.$$

There is often some confusion about the statements

(iv) $a \geqslant b$ (a is greater than *or* equal to b)

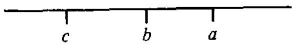
(v) $a \leqslant b$ (a is less than *or* equal to b).

To clear up this confusion, we note that the following are all true statements.

$$1 \geqslant 0;\ 3 \geqslant 2;\ 1 \geqslant 1;\ 2 \leqslant 3;\ -1 \leqslant 0;\ -3 \leqslant -3.$$

We assume four basic rules for the manipulation of inequalities. From these the other rules may be deduced.

(I) If $a > b$ and $b > c$, then $a > c$.



(II) If $a > b$ and c is any real number, then

$$a + c > b + c.$$

(III) If $a > b$ and $c > 0$, then $ac > bc$ (i.e. inequalities can be multiplied through by a *positive* factor).

(IV) If $a > b$ and $c < 0$, then $ac < bc$ (i.e. multiplication by a *negative* factor reverses the inequality).

1.5 *Example* If $a > 0$, prove that $a^{-1} > 0$.

Proof We argue by contradiction. Suppose that $a > 0$ but that $a^{-1} \leqslant 0$. It cannot be true that $a^{-1} = 0$ (since then $0 = 0.a = 1$). Hence

$$a^{-1} < 0.$$

By rule III we can multiply this inequality through by a (since $a > 0$). Hence

$$1 = a^{-1}.a < 0.a = 0.$$

But $1 < 0$ is a contradiction. Therefore the assumption $a^{-1} \leqslant 0$ was false. Hence $a^{-1} > 0$.

1.6 *Example* If x and y are positive, then $x < y$ if and only if $x^2 < y^2$.

Proof We have to show *two* things. First, that $x < y$ implies $x^2 < y^2$, and secondly, that $x^2 < y^2$ implies $x < y$.

(i) We begin by assuming that $x < y$ and try to deduce that $x^2 < y^2$. Multiply the inequality $x < y$ through by $x > 0$ (rule III). We obtain

$$x^2 < xy.$$

Similarly

$$xy < y^2.$$

But now $x^2 < y^2$ follows from rule I.

(ii) We now assume that $x^2 < y^2$ and try and deduce that $x < y$. Adding $-x^2$ to both sides of $x^2 < y^2$ (rule II), we obtain

$$y^2 - x^2 > 0$$

i.e. $$(y - x)(y + x) > 0. \qquad (1)$$

Since $x+y>0$, $(x+y)^{-1}>0$ (example 1.6). We can therefore multiply through inequality (1) by $(x+y)^{-1}$ to obtain

$$y-x>0$$

i.e. $x<y$.

(Alternatively, we could prove (ii) as follows. Assume that $x^2<y^2$ but that $x \geqslant y$. From $x \geqslant y$ it follows (as in (i)) that $x^2 \geqslant y^2$, which is a contradiction.)

1.7 *Example* Suppose that, for *any* $\epsilon>0, a<b+\epsilon$. Then $a \leqslant b$.

Proof Assume that $a>b$. Then $a-b>0$. But, for *any* $\epsilon>0$, $a<b+\epsilon$. Hence $a<b+\epsilon$ in the particular case when $\epsilon=b-a$. Thus

$$a<b+(a-b)$$

and so $a<a$.

This is a contradiction. Hence our assumption $a>b$ must be false. Therefore $a \leqslant b$.
(Note: The symbol ϵ in this example is the Greek letter *epsilon*. It should be carefully distinguished from the 'belongs to' symbol $\in$ and also from the symbol ξ which is the Greek letter *xi*.)

1.8 *Exercise*

(1) If x is any real number, prove that $x^2 \geqslant 0$. If $0<a<1$ and $b>1$, prove that

(i) $0<a^2<a<1$ (ii) $b^2>b>1$.

(2) If $b>0$ and $B>0$ and

$$\frac{a}{b}<\frac{A}{B},$$

prove that $aB<bA$. Deduce that

$$\frac{a}{b}<\frac{a+A}{b+B}<\frac{A}{B}.$$

(3) If $a>b$ and $c>d$, prove that $a+c>b+d$ (i.e. inequalities can be added). If, also, $b>0$ and $d>0$, prove that $ac>bd$ (i.e. inequalities between *positive* numbers can be multiplied).

(4) Show that each of the following inequalities may fail to hold even though $a>b$ and $c>d$.

(i) $a-c>b-d$

(ii) $\frac{a}{c} > \frac{b}{d}$

(iii) $ac > bd$.

What happens if we impose the extra condition that $b > 0$ and $d > 0$?

(5) Suppose that, for *any* $\epsilon > 0$, $a - \epsilon < b < a + \epsilon$. Prove that $a = b$.

(6) Suppose that $a < b$. Show that there exists a real number x satisfying $a < x < b$.

1.9 Roots

Let n be a natural number. The reader will be familiar with the notation $y = x^n$. For example, $x^2 = x.x$ and $x^3 = x.x.x$.

Our next assumption about the real number system is the following. Given any $y \geqslant 0$ there is exactly one value of $x \geqslant 0$ such that

$$y = x^n.$$

(Later on we shall see how this property may be deduced from the theory of continuous functions.)

If $y \geqslant 0$, the value of $x \geqslant 0$ which satisfies the equation $y = x^n$ is called the nth *root* of y and is denoted by

$$x = y^{1/n}.$$

When $n = 2$, we also use the notation $\sqrt{y} = y^{1/2}$. Note that, with this convention, it is always true that $\sqrt{y} \geqslant 0$. If $y > 0$, there are, of course, *two* numbers whose square is y. The positive one is $\sqrt{y}$ and the negative one is $-\sqrt{y}$. The notation $\pm\sqrt{y}$ means '$\sqrt{y}$ *or* $-\sqrt{y}$'.

If $r = m/n$ is a positive rational number and $y \geqslant 0$, we define

$$y^r = (y^m)^{1/n}.$$

If r is a negative rational, then $-r$ is a positive rational and hence y^{-r} is defined. If $y > 0$ we can therefore define y^r by

$$y^r = \frac{1}{y^{-r}}.$$

We also write $y^0 = 1$. With these conventions it follows that, if $y > 0$, then y^r is defined for all rational numbers r. (The definition of y^x when x is an irrational real number must wait until a later chapter.)

1.10 Quadratic equations

If $y > 0$, the equation $x^2 = y$ has two solutions. We denote the *positive* solution by $\sqrt{y}$. The *negative* solution is therefore $-\sqrt{y}$. We note again that there is no ambiguity about these symbols and that $\pm\sqrt{y}$ simply means '$\sqrt{y}$ or $-\sqrt{y}$'.

The general quadratic equation has the form

$$ax^2 + bx + c = 0$$

where $a \neq 0$. Multiply through by $4a$. We obtain

$$4a^2x^2 + 4abx + 4ac = 0$$

$$(2ax + b)^2 - b^2 + 4ac = 0$$

$$(2ax + b)^2 = b^2 - 4ac.$$

It follows that the quadratic equation has no real solutions if $b^2 - 4ac < 0$, one real solution if $b^2 - 4ac = 0$ and two real solutions if $b^2 - 4ac > 0$. If $b^2 - 4ac \geqslant 0$,

$$2ax + b = \pm\sqrt{(b^2 - 4ac)}$$

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}.$$

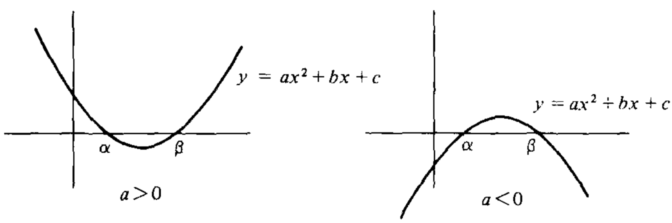
The roots of the equation $ax^2 + bx + c = 0$ are therefore

$$\alpha = \frac{-b - \sqrt{(b^2 - 4ac)}}{2a} \quad \text{and} \quad \beta = \frac{-b + \sqrt{(b^2 - 4ac)}}{2a}.$$

It is a simple matter to check that, for all values of x,

$$ax^2 + bx + c = a(x - \alpha)(x - \beta).$$

With the help of this formula, we can sketch the graph of the equation $y = ax^2 + bx + c$.



1.11 *Example* A nice application of the work on quadratic equations described above is the proof of the important *Cauchy–Schwarz inequality*. This asserts that, if $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are any real numbers, then

$$(a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 \leqslant (a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2).$$

Proof For any x,

$$0 \leqslant (a_1x+b_1)^2+(a_2x+b_2)^2+\ldots+(a_nx+b_n)^2$$
$$= (a_1^2+\ldots+a_n^2)x^2+2(a_1b_1+\ldots+a_nb_n)x+(b_1^2+\ldots+b_n^2)$$
$$= Ax^2+2Bx+C.$$

Since $y=Ax^2+2Bx+C \geqslant 0$ for *all* values of x, it follows that the equation $Ax^2+2Bx+C=0$ cannot have two (distinct) roots. Hence

$$(2B)^2-4AC \leqslant 0$$

i.e. $$B^2 \leqslant AC$$

which is what we had to prove.

1.12 *Exercise*

(1) Suppose that n is an *even* natural number. Prove that the equation $x^n=y$ has no solutions if $y<0$, one solution if $y=0$ and two solutions if $y>0$.

Suppose that n is an *odd* natural number. Prove that the equation $x^n=y$ always has one and only one solution.

Draw graphs of $y=x^2$ and $y=x^3$ to illustrate these results.

(2) Simplify the following expressions:

(i) $8^{2/3}$ (ii) $27^{-4/3}$ (iii) $32^{6/5}$.

(3) If $y>0$, $z>0$ and r and s are any rational numbers, prove the following:

(i) $y^{r+s}=y^ry^s$ (ii) $y^{rs}=(y^r)^s$ (iii) $(yz)^r=y^rz^r$.

(4) Suppose that $a>0$ and that α and β are the roots of the quadratic equation $ax^2+bx+c=0$ (in which $b^2-4ac>0$). Prove that $y=ax^2+bx+c$ is negative when $\alpha<x<\beta$ and positive when $x<\alpha$ or $x>\beta$. Show also (without the use of calculus) that $y=ax^2+bx+c$ achieves a minimum value of $c-b^2/4a$ when $x=-b/2a$.

(5) Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Their arithmetic mean A_n and harmonic mean H_n are defined by

$$A_n = \frac{a_1+a_2+\ldots+a_n}{n} \qquad H_n^{-1} = \frac{1}{n}\left\{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right\}.$$

Deduce from the Cauchy–Schwarz inequality that $H_n \leqslant A_n$.

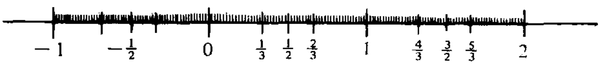
(6) Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any real numbers. Prove Minkowski's inequality, i.e.

$$\left\{\sum_{k=1}^{n}(a_k+b_k)^2\right\}^{1/2} \leqslant \left\{\sum_{k=1}^{n}a_k^2\right\}^{1/2} + \left\{\sum_{k=1}^{n}b_k^2\right\}^{1/2}.$$

For the case $n = 2$ (or $n = 3$) this inequality amounts to the assertion that the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides. Explain this.

1.13 Irrational numbers

In § 1.2 we mentioned the existence of irrational real numbers. That such numbers exist is by no means obvious. For example, one may imagine the process of marking all the rational numbers on a straight line. First one would mark the integers. Then one would move on to the multiples of $\frac{1}{2}$ and then to the multiples of $\frac{1}{3}$ and so on. Assuming that this program could ever be completed, one might very well be forgiven for supposing that there would be no room left for any more points on the line.



But our assumption about the existence of nth roots renders this view untenable. This assumption requires us to accept the existence of a positive real number x (namely $\sqrt{2}$) which satisfies $x^2 = 2$. If x were a rational number it would be expressible in the form

$$x = \frac{m}{n}$$

where m and n are natural numbers with no common divisor (other than 1). It follows that

$$m^2 = 2n^2$$

and so m^2 is even. This implies that m is even. (If m were odd, we should have $m = 2k + 1$. But then $m^2 = 4k^2 + 4k + 1$ which is odd.) We may therefore write $m = 2k$. Hence

$$4k^2 = 2n^2$$

$$n^2 = 2k^2.$$

Thus n is even. We have therefore shown that both m and n are divisible by 2. This is a contradiction and it follows that x cannot be rational, i.e. $\sqrt{2}$ must be an irrational real number.

Of course, $\sqrt{2}$ is not the only irrational number and the ability to extract nth roots allows us to construct many others. But it should not be supposed that all irrational numbers can be obtained in this way. It is not even true that every irrational number is a root of an equation of the form