Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

1272

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Commuting Nonselfadjoint Operators in Hilbert Space

Two Independent Studies

Springer-Verlag
Berlin Heidelberg New York London Paris Tokyo

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Mathematics Subject Classification (1980): 15, 35A07, 46, 47, 47D05, 47H06, 73, 81, 93

ISBN 3-540-18316-7 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-18316-7 Springer-Verlag New York Berlin Heidelberg

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COMMUTING NONSELFADJOINT OPERATORS AND COLLECTIVE MOTIONS OF SYSTEMS

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COMMUTING NONSELFADJOINT OPERATORS AND COLLECTIVE MOTIONS OF SYSTEMS

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Introduction

The spectral analysis of nonselfadjoint operators in the single operator case was developed during the fifties and sixties thanks to the efforts of many mathematicians. Then it was realized that this analysis forms a mathematical basis for the theory of open systems, interacting with the environment. In the light of the success of this theory, attempts have been made to create an analogous theory for several commuting operators, based on a generalization of the Nagy-Foias functional model to the case of several *independent* variables. It turned out, however, that this approach was ineffective. Moreover, the following generalization of the classical Cayley-Hamilton Theorem was obtained:

Two commuting operators with finite dimensional imaginary parts are connected, in generic case, by a certain algebraic equation. The degree of this equation does not exceed the dimension of the sum of ranges of imaginary parts.

This result shows that the genuine theory of commuting nonselfadjoint operators with finite dimensional imaginary parts is deeply connected with analytic functions on algebraic manifolds and it cannot be based only on functions of independent variables. Recent investigations of commuting operators was carried out in two directions. One of them deals with semigroups of shifts and it is presented in L.L. Waksman's monograph [13]. The other direction, which also is presented in this issue of "Lecture Notes" is based on operator colligations and collective motions of open systems [2-9]. In the case of collective motions all interacting systems distributed along the x-axis, form, at each given moment of time, one big spatial system. The corresponding input-output fields are compatible with internal collective states iff they satisfy certain partial differential equations of hyperbolic type. Actually, every given wave equation can be expressed as an external display of corresponding collective motions. In this way one can obtain, for instance, the Schrödinger, Klein-Gordon and Dirac equations. It turns out that the theory of commuting nonselfadjoint operators is deeply connected with the problem of wave dispersion in a medium: the above mentioned algebraic equation between commuting operators is nothing else but the corresponding dispersion law for the input-output waves. It is possible that the so called "quasi-particles" [1] can be described, at least from the mathematical point of view, as manifestations of collective motions. If the input field vanishes then the output manifestations of internal states behave like actions at a distance, decaying with respect to distance and depending also on time. If there exists a nonvanishing input wave, propagating along the x-axis, then quasi-particles are represented by a combined filed, consisting of the "ruling" input wave and of the all output manifestations, provoked by this wave.

§1. Single-Operator Colligations: Review of Basic Results

Let A be a bounded linear operator in a Hilbert space or in a finite dimensional space H. The closure G of the image of the difference $A-A^*$ is called the *nonhermitian* subspace of operator A. In many cases of interest this subspace is finite dimensional. The restriction of the type $\mathbf{dim}[\mathrm{range}(A-A^*)] < \infty$ plays an important role. As an example let us take the Newton-Leibnitz integral

$$(Af)(x) = i \int_0^x f(s)ds \ , \qquad (0 \leqslant x \leqslant \ell)$$

It is a nonselfadjoint operator in L_2 and the image of $2\,\mathrm{Im}(A)$:

$$\frac{1}{i}(A-A^*)f = \int_0^\ell f(s)ds \ , \qquad (0 \leqslant x \leqslant \ell)$$

is the one-dimensional subspace of all constant functions in the interval $0 \leqslant x \leqslant \ell$. Moreover [2,3] the Newton-Leibnitz integral can be characterized up to the unitary equivalence by the properties:

1) $\dim[\mathrm{range}(A-A^*)] = 1$
2) $\mathrm{spectrum}(A) = \{0\}$

This example shows that the number $n = \dim G$ can be helpful for the classification of nonselfadjoint operators.

Many problems of Mathematical Physics lead to equation of the form

$$i\frac{\partial f}{\partial t} + Af = 0 \ , \tag{1}$$

where $f(t) \in H$ is a state of the corresponding system. In physical applications the energy (or the number of particles) in the state f

is proportional to the scalar product (f,f). If $A=A^*$ then the scalar product $(f(t),f(t))$ is a constant and in this case the system is said to be closed. In general the system interacts with environment and the corresponding operator A is not selfadjoint. It is always possible to represent the difference $A-A^*$ in the form

$$\frac{1}{i}(A-A^*) = \Phi^*\sigma\Phi \tag{2}$$

where $\Phi: H \to E$ is a linear mapping of H into a new Hilbert (or finite dimensional) space E, σ is a bounded *selfadjoint* operator in E and $\phi^*: E \to H$ is the adjoint of Φ:

$$(\Phi f,u) = (f,\Phi^* u) , \qquad (f \in H,\ u \in E)$$

One of the representations of the form (2) can be derived in the following way:

$$\frac{1}{i}(A-A^*) = P_G\sigma P_G ,$$

where G is the nonhermition subspace, P_G is the orthogonal projection onto G,

$$\sigma = \frac{1}{i}(A-A^*)|_G$$

and $E=G$, $\Phi=P_G$.

The set $X = (A,H,\phi,E,\sigma)$ which satisfies the condition (2) is called a *colligation*. In the following we assume that subspace E is finite dimensional: $\dim E = n < \infty$. The space H is called the *internal* space of the colligation and E - the *coupling* space.

With every colligation we associate an open system which is defined by equations of the form

$$i\frac{df}{dt} + Af = \phi^*\sigma[u(t)] \tag{3}$$

$$v(t) = u(t) - i\phi[f(t)] \tag{4}$$

where $v(t)$, $v(t) \in E$ are an input, and an output respectively. These relations have not been chosen arbitrarily. It is easy to check that for such a system the following law of metric (energy, number of particles) balance holds:

$$\frac{d}{dt}(f,f) = (\sigma u,u) - (\sigma v,v) , \tag{5}$$

where $(\sigma u,u)$ and $(\sigma v,v)$ can be interpreted as metric flows through the input and the output respectively.

Proof of the metric balance formula:

Using (3) and (4) we obtain

$$\frac{d}{dt}(f,f) = (\frac{df}{dt}, f) + (f, \frac{df}{dt}) = (iAf-i\phi^*\sigma(u),f) + (f,iAf-i\phi^*\sigma(u)) =$$

$$= (i(A-A^*)f,f) - i(\sigma(u),\Phi f) + i(\Phi f,\sigma(u))$$

and

$$(\sigma u,u) - (\sigma v,v) = (\sigma u,u) - (\sigma(u-i\Phi(f)),u-i\Phi(f)) =$$

$$= i(\Phi(f),\sigma(u)) - i(\sigma(u),\Phi(f)) - (\Phi^*\sigma\Phi(u),u)$$

The colligation condition (2) implies (5).

Let us consider a solution of equation (3) of the form

$$u(t) = u_0 e^{i\lambda t}, \quad f(t) = f_0 e^{i\lambda t}, \quad v(t) = v_0 e^{i\lambda t} \tag{6}$$

It is easy to see that

$$f_0 = (A-\lambda I)^{-1}\phi^* u_0 \tag{7}$$

$$v_0 = S(\lambda)u_0 , \tag{8}$$

where

$$S(\lambda) = I-i\phi(A-\lambda I)^{-1}\phi^*\sigma(A) \tag{9}$$

and λ is a regular point of the resolvent.

Operator-function $S(\lambda)$ is said to be the *characteristic* (transfer) function of colligation (A,H,ϕ,E,σ). From (5) it follows for solution (6) that

$$i(\lambda-\lambda^*)(f_0,f_0) = (\sigma u_0,u_0)-(\sigma v_0,v_0) \tag{10}$$

This formula implies the following

Theorem 1

The characteristic function $S(\lambda)$ *has the following properties:*

$$(\mathrm{Im}\lambda)[S^*(\lambda)\sigma S(\lambda)-\sigma] > 0 , \qquad (\mathrm{Im}\lambda \neq 0) \tag{11}$$

$$S^*(\lambda)\sigma S(\lambda) = \sigma , \qquad (\mathrm{Im}\lambda = 0) \tag{12}$$

It can be proved the following important [2]

Factorization Theorem

Let $H = H_0 \supset H_1 \supset H_2 \supset \dots H_{m-1} \supset H_m = 0$ *be a chain of invariant subspaces of operator* A *and* $H_k^\perp = H_k \ominus H_{k-1}$. Then the characteristic function $S(\lambda)$ is a product

$$S(\lambda) = S_m(\lambda)S_{m-1}(\lambda) \dots S_2(\lambda)S_1(\lambda) \tag{13}$$

where $S_k(\lambda)$ $(k=1,2,\dots,m)$ is the characteristic function of the colligation

$$X_k = (A|_{H_k^\perp},H_k^\perp,P_k\Phi,E,\sigma) \tag{14}$$

(P_k is the orthogonal projection on $H_k^\perp$)

The colligation $X_k = P_k(X)$ is said to be the *projection* of X onto $H_k^{\perp}$. The colligation X is said to be the coupling of the projections X_k: $X = X_m \vee X_{m-1} \ldots \vee X_1$. To the coupling of colligations there correspondes the *chain coupling* [2,9] of corresponding systems: the output of each link coincides with the input of the next one.

It is easy to check that the subspace

$$\hat{H} = \text{span}\{A^k \phi^*(E)\} = \text{span}\{A^{*k}\phi^*(E)\}\ , \qquad (0 \leq k < \infty)$$

reduces A and that the restriction of A to the orthogonal complement $H_0 = H \ominus \hat{H}$ is selfadjoint. The subspace $\hat{H}$ is said to be the *principal* subspace of the colligation.

Theorem 2 [2]. *Let the space* E *and the selfadjoint operator* σ *be given. Assume that* $\det(\sigma) \neq 0$. *Then the characteristic function* $S(\lambda)$ *determines a corresponding colligation up to an unitary transformation of its principal component. The class* $\Omega(\sigma)$: An operator-function $W(\lambda)$: $E \to E$ is said to be a function of the class $\Omega(\sigma)$ if it has the following properties:

1) $W(\lambda)$ is a meromorphic function in the open upper half plane $\text{Im}\lambda > 0$
2) $W(\lambda)$ is holomorphic in a neighbourhood $|\lambda| > a$ of $\lambda = \infty$ and $W(\infty) = I$.
3) $W^*(\lambda)\sigma W(\lambda) > \sigma \quad (\text{Im}\lambda > 0)$ (15)
4) $W^*(\lambda)\sigma W(\lambda) = \sigma \quad (\text{Im}\lambda = 0)$ (16)

The following theorem holds [2]:

Theorem 3

Let $\sigma = \sigma^*$ *be a given invertible operator in* E. *A given function* $W(\lambda)$ *belongs to the class* $\Omega(\sigma)$ *iff* $W(\lambda)$ *is the characteristic function of some colligation* $X = (A,H,\phi,E,\sigma)$.
If, for instance, $\dim E = 1$, then $S(\lambda)$ is a scalar function and we can assume that $\sigma=1$. In this case the characteristic function $S(\lambda)$ can be represented in the following form [2,9]:

$$S(\lambda) = \prod_{k=1}^{N} \frac{\lambda - \lambda_k^*}{\lambda - \lambda_k} \cdot \exp\left[i \int_0^{\ell} \frac{ds}{\lambda - \alpha(s)} \right]\ , \qquad 0 \leq N \leq \infty \tag{17}$$

$(\text{Im}\lambda_k > 0,\ \Sigma\ \text{Im}\lambda_k < \infty)$, where $\alpha(s)$ is a real nondecreasing function. The corresponding operator $Af=f'$ can be representing (up to a unitary transformation) in the space $\ell_2 \oplus L_2$ in the following triangular form

$$f_k' = \lambda_k f_k + i \sum_{j=k+1}^{N} \beta_k \beta_j f_j + i\beta_k \int_0^{\ell} f(s)ds, \quad (j=1,2,\ldots,N)$$

$$f'(x) = \alpha(x)f(x) + i \int_0^{x} f(s)ds, \quad (0 \leq x \leq \ell) \tag{18}$$

where $\beta_k = (2 \operatorname{Im} \lambda_k)^{\frac{1}{2}}$. If $\dim H < \infty$, then $\ell=0$ and $N = \dim H$. If $\operatorname{spectrum}(A) = \{0\}$ then $N=0$ and $\alpha(x) \equiv 0$. In this case the triangular model turns into the Newton-Leibnitz operator

$$(Af)(x) = i \int_0^{x} f(s)ds, \quad 0 \leq x \leq \ell. \tag{19}$$

The characteristic function in this case is

$$S(\lambda) = \exp(i \frac{\ell}{\lambda}) \tag{20}$$

For the generalization of formulas (17), (18) to the case $n > 1$ see [2].

§2. Commutative Colligations and Collective Motions

Let (A,B) be a pair of linear bounded operators in H. Define the subspaces

$$G_A = \overline{(A-A^*)H}, \quad G_B = \overline{(B-B^*)H}, \quad G = G_A + G_B \tag{21}$$

The subspace G is called the nonhermitian subspace of the pair (A,B). We assume in the following that $\dim G = n < \infty$. Let us consider, for example, the operators

$$(Af)(x) = i \int_0^{x} f(s)ds$$

$$(f \in L_2, \ 0 \leq x \leq \ell)$$

$$(Bf)(x) = \int_0^{x} (s-x)f(s)ds$$

It is easy to see that $B = A^2$ and

$$\frac{1}{i}(A-A^*)f = \int_0^{\ell} f(s)ds = (f,1)1$$

$$\frac{1}{i}(B-B^*)f = i \int_0^{\ell} (x-s)f(s)ds = i(f,1)x - i(f,x)1$$

Hence $G = \{c_1 + c_2 x\}$ is two dimensional

Definition. Let H,E be Hilbert spaces, (A,B)-bounded linear operators in H; $\sigma(A)$, $\sigma(B)$-bounded selfadjoint operators in E, Φ - a bounded linear mapping of H into E. A set

$$X = (A,B;H,\Phi,E;\sigma(A),\sigma(B)) \tag{22}$$

is said to be a colligation if

$$\frac{1}{i}(A-A^*) = \Phi^*\sigma(A)\Phi, \quad \frac{1}{i}(B-B^*) = \Phi^*\sigma(B)\Phi \tag{23}$$

If range $\Phi = E$ and $\operatorname{Ker}\sigma(A) \cap \operatorname{Ker}\sigma(B) = 0$ the colligation is called a *strict* colligation. A colligation is said to be *commutative* if $AB=BA$. An arbitrary given pair (A,B) can be embedded in a strict colligation with $E=G$, $\Phi=P_G$, $\sigma(A) = \frac{1}{i}(A-A^*)|_G$, $\sigma(B) = \frac{1}{i}(B-B^*)|_G$.

If $AB=BA$ then the ranges of selfadjoint operators

$$\frac{1}{i}(AB^*-BA^*) = i[(B-B^*)A^*-(A-A^*)B^*] \tag{24}$$

$$\frac{1}{i}(B^*A-A^*B) = i[(B-B^*)A-(A-A^*)B] \tag{25}$$

belong to the nonhermitian subspace

$$G = \operatorname{range}(A-A^*)+\operatorname{range}(B-B^*) \tag{26}$$

Remark. If X is strict then $\Phi^*E=G$, $\Phi G=E$ and $\dim E = \dim G$. Indeed, relation (23) imply that $\operatorname{range}(A-A^*) = \operatorname{range}\Phi^*\sigma(A)$, $\operatorname{range}(B-B^*) = \operatorname{range}\Phi^*\sigma(B)$ and $G = \Phi^*(\operatorname{range}\sigma(A)+\operatorname{range}\sigma(B)) = \Phi^*E$. If one assumes that $(\Phi G,U_0)=0$ then $(G,\Phi^*U_0)=0$ and $\Phi^*U_0=0$. Hence $(\Phi H,U_0) = (H,\Phi^*U_0) = 0$ and $U_0 = 0$.

Corollary. If X is strict and commutative then there are two selfadjoint operators γ and $\tilde{\gamma}$ in E, satisfying the following conditions:

$$\frac{1}{i}(AB^*-BA^*) = \Phi^*\gamma\Phi \tag{27}$$

$$\frac{1}{i}(B^*A-A^*B) = \Phi^*\tilde{\gamma}\Phi\ , \tag{28}$$

where $\Phi_0 = \Phi|G$.

Remark. The equalities (24) and (23) imply

$$2\operatorname{Im}(AB^*) = \Phi^*[\sigma(A)\Phi B^*-\sigma(B)\Phi A^*]\ ,$$

$$2\operatorname{Im}(B^*A) = \Phi^*[\sigma(A)\Phi B-\sigma(B)\Phi A]$$

Hence for strict colligations we obtain:

$$\sigma(A)\Phi B^*-\sigma(B)\Phi A^* = \gamma\Phi$$

$$\sigma(A)\Phi B-\sigma(B)\Phi A = \tilde{\gamma}\Phi\ ,$$

Then substituting and using (23) we obtain

$$i[\sigma(A)\Phi\Phi^*\sigma(B)-\sigma(B)\Phi\Phi^*\sigma(A)]\Phi = (\tilde{\gamma}-\gamma)\Phi$$

Hence, for strict colligation the relation

$$\tilde{\gamma} - \gamma = i[\sigma(A)\Phi\Phi^*\sigma(B)-\sigma(B)\Phi\Phi^*\sigma(A)]$$

is valid.

Operators γ, $\tilde{\gamma}$ play an important role but the condition $\Phi(H) = E$ is too restrictive: the projection of a strict colligation on a subspace is not always strict. Moreover, in the simplest case when $\dim H = 1$, colligations are not strict unless $\dim E = 1$. To overcome these difficulties we use the notion of a regular colligation, introduced by N. Kravitsky [5].

Definition. A commutative colligation is said to be *regular* if there exists a selfadjoint operator γ in E such that the equality

$$\sigma(A)\Phi B^* - \sigma(B)\Phi A^* = \gamma\Phi \tag{29}$$

is valid. Let us define $\tilde{\gamma}$ as follows:

$$\tilde{\gamma} = \gamma + i[\sigma(A)\Phi\Phi^*\sigma(B) - \sigma(B)\Phi\Phi^*\sigma(A)] \tag{29'}$$

It is easy to check that $\tilde{\gamma}$ satisfies the following condition

$$\sigma(A)\Phi B - \sigma(B)\Phi A = \tilde{\gamma}\Phi \tag{30}$$

Analogously from (30), (29') it follows (29). It is evident that strict colligations are regular. The operators γ and $\tilde{\gamma}$ are defined uniquely for strict colligations but for regular colligations it is not so. We will include operators γ and $\tilde{\gamma}$ in the notation of a regular colligation

$$X = (A,B;H,\Phi,E;\sigma(A),\sigma(B),\gamma,\tilde{\gamma}) ,$$

where γ and $\tilde{\gamma}$ are connected by the so called "linkage" equality (29'). They satisfy the regularity conditions (29), (30). The regularity conditions can be written in a determinantal form

$$\begin{vmatrix} \sigma(A)\Phi, & A^* \\ \sigma(B)\Phi, & B^* \end{vmatrix} = \gamma\Phi , \qquad \begin{vmatrix} \sigma(A)\Phi, & A \\ \sigma(B)\Phi, & B \end{vmatrix} = \tilde{\gamma}\Phi \tag{31}$$

and the linkage equation has the form

$$\begin{vmatrix} \sigma(A)\Phi, & \Phi^*\sigma(A) \\ \sigma(B)\Phi, & \Phi^*\sigma(B) \end{vmatrix} = \frac{1}{i}(\tilde{\gamma}-\gamma) \tag{32}$$

Let us assume now that an input, a state and an output depend also on a spatial coordinate x $(x_0 \leqslant x \leqslant x_1)$.

Definition. An input $u(t,x)$, a state $f(t,x)$ and an output $v(t,x)$ are said to be a *collective* input, state and output respectively if they satisfy equations of the form:

$$i\frac{\partial f}{\partial t} + Af = \Phi^*\sigma(A)[u(t,x)] , \tag{33}$$

$$i\frac{\partial f}{\partial x} + Bf = \Phi^*\sigma(B)[u(t,x)] \tag{34}$$

$$v(t,x) = u(t,x) - i\Phi[f(t,x)] \ , \tag{35}$$

where

$$X = (A,B;H,\Phi,E;\sigma(A),\sigma(B)) \tag{36}$$

is a colligation, i.e.

$$\begin{aligned} \frac{1}{i}(A-A^*) &= \Phi^*\sigma(A)\Phi \\ \frac{1}{i}(B-B^*) &= \Phi^*\sigma(B)\Phi \ . \end{aligned} \tag{37}$$

So, at the arbitrary fixed point x $(x_0 \leqslant x \leqslant x_1)$ the equation (33) defines a temporal system. The input and the internal state depend also on x. In the case of collective motions all temporal systems, distributed along the x-axis, at the same moment of time behave like one big spatial system: the equation (34) defines a connection between the internal states at different points of the x-axis. For collective motions the following laws of metric balance hold:

$$\frac{\partial}{\partial t}(f,f) = (\sigma(A)u,u) - (\sigma(A)v,v) \tag{38}$$

$$\frac{\partial}{\partial x}(f,f) = (\sigma(B)u,u) - (\sigma(B)v,v) \tag{39}$$

Assuming that equations (33) and (34) are consistent for arbitrary initial conditions $f(t_0,x_0)=f_0$ we conclude that $AB=BA$. On the line

$$t = \xi_1\tau+t_0 \ , \qquad x = \xi_2 c\tau + x_0 \tag{40}$$

where c denotes the light speed in the vacuum, equations (33), (34) imply

$$i\,\frac{df}{d\tau} + (\vec{\xi}\cdot\vec{A})f = \Phi^*\sigma(\vec{\xi}\cdot\vec{A})(u) \ , \tag{41}$$

$$v = u - i\Phi(f) \ , \tag{42}$$

where $\vec{\xi}\cdot\vec{A} = \xi_1 A_1 + \xi_2 A_2$, $A_1 = A$, $A_2 = cB$.

The system (41) in the direction $\vec{\xi} = (\xi_1,\xi_2)$ corresponds to the motion $x = V\cdot(t-t_0)+x_0$ of the point x along the x-axis with the constant velocity $V = \frac{\xi_2}{\xi_1}c$. If $\xi_1=1$, $\xi_2=0$ then $V=0$ and $x=x_0$. In this case we obtain the temporal system at fixed point $x=x_0$. If $\xi_1=0$, $\xi_2=1$ then $V=\infty$ and $t=t_0$, which corresponds to the spatial system at fixed moment of time $t=t_0$. Thus, in the case of collective motions there exists a family of open systems related to motions $x = Vt+x_0$ along the x-axis. In particular, the spatial system corresponds to the *infinite* speed of the point x, which is an *immediate action at a distance: the spatial system defines a solid frame for all possible motions of temporal systems, distributed along* *the* x-*axis*.

Theorem 4

Assume that X *is a strict commutative colligation. Then an input* u(t,x) *is a collective input if and only if* u(t,x) *satisfies the* PDE

$$\sigma(B)\frac{\partial u}{\partial t} = \sigma(A)\frac{\partial u}{\partial x} - i\gamma u \tag{43}$$

The corresponding output satisfies the equation

$$\sigma(B)\frac{\partial v}{\partial t} = \sigma(A)\frac{\partial v}{\partial x} - i\tilde{\gamma}v \tag{44}$$

Proof. We use integrability conditions

$$\frac{\partial^2 f}{\partial t\partial x} = \frac{\partial^2 f}{\partial x\partial t}$$

Equations (33), (34) imply

$$\frac{\partial}{\partial x}(Af-\Phi^*\sigma(A)u) = \frac{\partial}{\partial t}(Bf-\Phi^*\sigma(B)u)$$

Hence

$$A\frac{\partial f}{\partial x} - B\frac{\partial f}{\partial t} = \Phi^*[\sigma(A)\frac{\partial u}{\partial x} - \sigma(B)\frac{\partial u}{\partial t}] \tag{45}$$

Equations (33), (34) also imply

$$A\frac{\partial f}{\partial x} - B\frac{\partial f}{\partial t} = i[B\Phi^*\sigma(A)-A\Phi^*\sigma(B)]u$$

Using (29) we obtain

$$A\frac{\partial f}{\partial x} - B\frac{\partial f}{\partial t} = i\Phi^*\gamma \tag{46}$$

Equalities (45), (46) imply

$$\Phi^*[\sigma(A)\frac{\partial u}{\partial x} - \sigma(B)\frac{\partial u}{\partial t} - i\gamma u] = 0$$

which implies for strict colligations the equation (43). For $v=u-i\Phi f$ we obtain

$$\sigma(A)\frac{\partial v}{\partial x} - \sigma(B)\frac{\partial v}{\partial t} - i\tilde{\gamma}v =$$

$$= \sigma(A)\frac{\partial u}{\partial x} - \sigma(B)\frac{\partial u}{\partial t} - i\sigma(A)\Phi\frac{\partial f}{\partial x} + i\sigma(B)\Phi\frac{\partial f}{\partial t} - i\tilde{\gamma}u-\tilde{\gamma}\Phi f =$$

$$= i(\gamma-\tilde{\gamma})u + [\sigma(A)\Phi B-\sigma(B)\Phi A]f - [\sigma(A)\Phi\Phi^*\sigma(B)-\sigma(B)\Phi\Phi^*\sigma(A)]u-\tilde{\gamma}\Phi f .$$

Using (30) in the form

$$\sigma(A)\Phi B-\sigma(B)\Phi A = \tilde{\gamma}\Phi \tag{47}$$

and

$$\tilde{\gamma}-\gamma = i(\sigma(A)\Phi\Phi^*\sigma(B)-\sigma(B)\Phi\Phi^*\sigma(A)) \tag{48}$$

we obtain

$$\sigma(A)\frac{\partial v}{\partial x} - \sigma(B)\frac{\partial v}{\partial t} - i\tilde{\gamma}v = i(\gamma-\tilde{\gamma})u + \tilde{\gamma}\Phi u + i(\tilde{\gamma}-\gamma)u - \tilde{\gamma}\Phi f = 0$$

Remarks: 1) Theorem 4 implies that γ corresponds to the input and $\tilde{\gamma}$ corresponds to the output in a natural way. We will denote $\gamma=\gamma^{in}$ and $\tilde{\gamma}=\gamma^{out}$. 2) If $\sigma(B) > 0$ then PDE (43), (44) are of *hyperbolic* type. Let us consider collective motions of the form

$$u = u_0 e^{i(\lambda t+\mu x)}, \quad f = f_0 e^{i(\lambda t+\mu x)}, \quad v = v_0 e^{i(\lambda t+\mu x)} \tag{49}$$

Theorem 4 implies that u and v are solutions of equations (43) and (44) respectively. Hence u_0 and v_0 are solutions of algebraic equations

$$(\lambda\sigma(B)-\mu\sigma(A)+\gamma^{in})u_0 = 0 \tag{50}$$

$$(\lambda\sigma(B)-\mu\sigma(A)+\gamma^{out})v_0 = 0 \tag{51}$$

Later we will prove the following important equality:

$$\det(\lambda\sigma(B)-\mu\sigma(A)+\gamma^{in}) \equiv \det(\lambda\sigma(B)-\mu\sigma(A)+\gamma^{out}) \tag{52}$$

The polynomial

$$D(\lambda,\mu) = \det(\lambda\sigma(B)-\mu\sigma(A)+\gamma^{in}) \tag{53}$$

is said to be the *discriminant* of the colligation. The corresponding algebraic curve

$$\Gamma = \{(\lambda,\mu) \in C_2,\ D(\lambda,\mu) = 0\}$$

is said to be the *discriminant curve*.

The subspaces

$$E^{in}(M) = \mathrm{Ker}(\lambda\sigma(B)-\mu\sigma(A)+\gamma^{in}), \tag{54}$$

$$E^{out}(M) = \mathrm{Ker}(\lambda\sigma(B)-\mu\sigma(A)+\gamma^{out}) \tag{55}$$

where $M = (\lambda,\mu)$ is an arbitrary point of the curve Γ are said to be the joint input and the joint output subspaces respectively. Equations (50), (51) imply that $u_0 \in E^{in}(M)$, $v_0 \in E^{out}(M)$. In the case $\sigma(B) > 0$ to each real value μ there correspond n real roots $\lambda_1(\mu), \ldots, \lambda_n(\mu)$ of the equation

$$D(\lambda,\mu) = 0 \tag{56}$$

Hence, to each real value μ there correspond n plane collective waves of the form (49).

§3. Characteristic (Transfer) Functions

Let us consider special motions

$$u = u_0 e^{iz\tau}, \quad f = f_0 e^{iz\tau}, \quad v = v_0 e^{iz\tau} \tag{57}$$