Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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These notes are concerned with seminormal operators on a Hilbert space. In the past decade several major results on this class of operators have been obtained, some of which appear mystifying and have created stirs of interest in the area. These results have come from (at least) five different sources and this makes it somewhat difficult to appreciate what is happening. The aim of these notes is to paint a reasonably self-contained picture of some of the developments in the area of seminormal operators which have occurred during the last ten years.

The fact that seminormal operators are interesting became obvious in 1970 when C. R. PUTNAM [5] established that the planar Lebesgue measure of the spectrum of any non-normal seminormal operator is positive. Perhaps just as deep is the result obtained in 1973 by C. A. BERGER and B. I. SHAW [1] which shows that any hyponormal operator with a cyclic vector has a trace class self-commutator. While these two developments were taking place R. W. CAREY and J. D. PINCUS were studying an invariant (referred to as the principal function) for operators having a trace class self-commutator. This invariant arose as a two dimensional analogue of the phase shift from the theory of trace class perturbations of self-adjoint operators. HELTON and HOWE [1], in an attempt to understand the work of Carey and Pincus, began studying the star-algebra generated by polynomials in an operator that has a trace class self-commutator. These authors introduced a tracial bilinear form on this algebra which they represented via integration up against a signed measure on the plane. Pincus immediately verified this measure has derivative equal to the principal function. Independently, these authors have used these invariants to describe a fairly complete spectral theory for seminormal operators. A most interesting result concerning these matters is the recent work of C. A. BERGER [1] which shows that the size of the principal function for a hyponormal operator can give information concerning cyclic vectors. That such information was carried in the principal function was conjectured by Helton and Howe. In another direction, J. G. STAMPFLI [8] has produced an interesting dichotomy in the local spectral theory of seminormal operators. Stampfli has shown that local spectral subspaces of hyponormal operators are always closed (but possibly trivial), whereas in the cohyponormal case the local spectral spaces are always non-trivial. It is the above mentioned results of these authors that occupies the major portion of these notes.

The notes are organized as follows.

Chapter 1 is mainly concerned with the local spectral theory of seminormal operators. Examples and simple applications of local spectral theory are presented.

Chapter 2 contains a "singular integral" model for seminormal operators. This model plays an important role in the remaining portion of the notes. In this chapter the model is used to make transparent a pair of self-adjoint commutator inequalities of Putnam [2] and Kato[2].

Chapter 3 contains what its title describes. First, we derive Putnam's inequality which establishes that the planar Lebesgue measure of the spectrum of a non-normal seminormal operator is positive. Secondly, we derive the result of Berger and Shaw [1] which establishes that a hyponormal operator with a cyclic vector has a trace class self-commutator.

Chapter 4 presents a discussion of the phase shift of M.G. Krein [1]. This phase shift arises in connection with trace class perturbations of self-adjoint operators. The existence and properties of the phase shift are crucial to our proof of the existence of the principal function. Several remarks concerning the phase shift are presented which are intended to give the reader a better feeling for the principal function.

Chapter 5 contains a brief study of nearly normal operators. For a portion of this chapter we restrict to the seminormal case. This provides several simplifying advantages. One such advantage is the ease with which we can compute the principal function for singular integral representations of seminormal operators. The finale of the notes is a result of Berger [1] relating the size of the principal function to the existence of cyclic vectors.

A general "thank you" is offered to my colleagues, students, and friends who have influenced the writing of these notes. More specific thanks are given to Ann Ware and Dianne Byrd for their careful typing of this work, and to Tom Howe for a final proofreading. Finally, a special thanks to my wife Carolyn for her constant support.

K.C.

Athens, Georgia
Spring 1979 .

CONTENTS

CHAPTER I

SEMINORMAL OPERATORS

In this chapter, we set down some of the basic properties of seminormal operators. A fairly complete description of the local spectral theory for seminormal operators is presented. Examples are constructed and spectral theory is illuminated with these examples.

1. Definitions and Basic Properties.

The notation $\mathcal{H}$ is reserved for a complex separable Hilbert space with inner product (,). The term operator indicates a bounded linear operator on $\mathcal{H}$. Operators will be denoted by capital letters A, B, C, ... The full algebra of operators on $\mathcal{H}$ is denoted by $L(\mathcal{H})$. As is customary, if A, B are operators on $\mathcal{H}$, then their commutator is denoted by $[A,B] = AB - BA$. The commutator $[A^*,A] = A^*A - AA^*$ is referred to as the self-commutator of the operator A.

As the reader surely knows, an operator $N \in L(\mathcal{H})$ is called normal when N commutes with N^*. Equivalently, N is normal when the self-commutator $[N^*,N]$ is zero. A generalization of normality is the following: An operator $S \in L(\mathcal{H})$ is called seminormal in case its self-commutator $D = [S^*,S]$ is semidefinite. In the case $D \geq 0$ the operator S is called hyponormal and when $D \leq 0$ the operator S is called cohyponormal. In spite of the fact that the adjoint of the class of hyponormal operators is merely the class of cohyponormal operators, there are several important stages where the development of the classes is entirely different. Examples of these differences appear in local spectral theory and in the study of cyclic vectors.

For any $A \in L(\mathcal{H})$ and $\alpha, \beta \in \mathbb{C}$

$$[(\alpha A+\beta)^*, \alpha A+\beta] = |\alpha|^2 [A^*, A] .$$

Consequently, $\alpha S + \beta$ will be seminormal whenever S is seminormal. This last remark remains valid if in both instances the word "seminormal" is replaced by either "hyponormal" or "cohyponormal".

Let $A \in L(\mathcal{H})$ and $z \in \mathbb{C}$. The notation $A_z = A - zI$ will be used. To avoid confusion, we indicate that $A_z^* = (A_z)^* = (A^*)_{\bar{z}}$. Let $A = X + iY$ be the <u>Cartesian form</u> of the operator A, so that $X = \frac{1}{2}[A+A^*]$ and $Y = \frac{1}{2i}[A - A^*]$. Write $z = x + iy$ in Cartesian form. Easy computations show

$$A_z^* A_z = X_x^2 + Y_y^2 + i[XY-YX] \tag{1.1}$$

$$A_z A_z^* = X_x^2 + Y_y^2 - i[XY-YX] \tag{1.2}$$

$$[A_z^*, A_z] = 2i[XY-YX]. \tag{1.3}$$

The identities (1.1)-(1.3) will be used freely in the sequel. Identity (1.3) provides an obvious connection between seminormal operators and pairs of self-adjoint operators with semidefinite commutators.

The following simple lemma will be used.

<u>Lemma 1.1</u>. Let A and B be in $L(\mathcal{H})$. In order that the inequality

$$\|Af\| \le \|Bf\|$$

hold for every $f \in \mathcal{H}$, it is necessary and sufficient that $A = KB$ for some contraction operator K.

Proof. The sufficiency of the condition is clear. If the inequality $\|Af\| \le \|Bf\|$ holds, then we define $Kg = Af$, whenever $g = Bf$. This defines K on the range of the operator B, moreover, $\|Kg\| \le \|g\|$, for g in this range. Thus K extends to a contraction on the closure of the range of B. Setting K to be zero on the orthogonal complement of the range of B provides a contraction operator on $\mathcal{H}$ such that A = KB. The lemma is proved.

The following proposition describes three equivalent formulations of hyponormality:

<u>Proposition 1.1</u>. Let $H \in L(\mathcal{H})$. The following statements are equivalent:

(i) $H^*H - HH^* \geq 0$

(ii) $\|H^*f\| \leq \|Hf\|$, for all $f \in \mathcal{H}$.

(iii) $H^* = KH$, for some contraction operator K.

Proof. Statements (i) and (ii) are obviously equivalent. The equivalence of (ii) and (iii) follows immediately from Lemma 1.1. The proposition is proved.

The following properties of hyponormal operators follow quickly from Proposition 1.1.

1°. Let H be hyponormal. Then we have the inclusion

$$\ker H \subset \ker H^*, \tag{1.4}$$

where for $A \in L(\mathcal{H})$, we have used the notation ker A to indicate the <u>kernel</u> of the operator A. Note, in particular, the inclusion (1.4) shows that ker H is a reducing subspace for the hyponormal operator H.

2°. Let H be hyponormal. Then we have the inclusion

$$R(H) \subset R(H^*), \tag{1.5}$$

where for $A \in L(\mathcal{H})$, we have used the notation R(A) for the <u>range</u> of the operator A. The inclusion (1.5) follows by taking adjoints in statement (iii) of Proposition 1.1 .

From 2° we learn the following. If the equation

$$(H-\lambda)f = \phi$$

has a solution $f = \phi_\lambda$, then the equation

$$(H^*-\bar{\lambda})f = \phi$$

has a solution $f = \Psi_{\bar{\lambda}} = [K(\lambda)]^*\phi_\lambda$ satisfying

$$\|\Psi_{\bar{\lambda}}\| \leq \|\phi_\lambda\| ,$$

where $K(\lambda)$ denotes some contraction satisfying $H_\lambda^* = K(\lambda)H_\lambda$. We note that $K(\lambda)$ is not uniquely determined (this can be remedied by insisting that $K(\lambda)$ be zero on $[R(H_\lambda)]^\perp$). Further, the dependence of $K(\lambda)$ on the parameter λ is not simple.

3°. Let H be an invertible hyponormal operator. The operator H^{-1} is also hyponormal. In fact, if $H^* = KH$, then $K = H^*H^{-1}$. Thus $K^* = (H^{-1})^*H$, which provides the equation $(H^{-1})^* = K^*H^{-1}$. The hyponormality of H^{-1} follows from Proposition 1.1. For another proof of this remark, we could compute

$$[(H^{-1})^*,H^{-1}] = H^{-1}(H^{-1})^*[H^*,H](H^{-1})^*H^{-1} .$$

The latter identity can be used to show H^{-1} is hyponormal. This computation also shows that the ranks of the self-commutators of H and H^{-1} are equal. Similar remarks can be made for seminormal and cohyponormal operators.

The following lemma will be used twice in the sequel. Its simple proof is left as an exercise:

Lemma 1.2. Let $\{a_n\}_{n=1}^\infty$ be a sequence of positive numbers which satisfy the relations

$$a_1^2 \leq a_2 \quad \text{and} \quad a_n^2 \leq a_{n-1}a_{n+1}, \; n = 2,3,\ldots \quad .$$

Then

$$a_1^n \leq a_n, \ n = 1,2, \ldots \quad .$$

The notation $\sigma(T)$ will be used for the <u>spectrum</u> of an operator $T \in L(\mathcal{H})$. The <u>spectral radius</u> of T will be denoted by $r_{sp}(T)$. Thus

$$r_{sp}(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}.$$

The following proposition gives an example of one of the properties which seminormal operators share with normal operators:

<u>Proposition 1.2</u>. Let S be a seminormal operator in $L(\mathcal{H})$. Then

$$\|S\| = r_{sp}(S) \ .$$

Proof. We will employ the identity

$$r_{sp}(S) = \overline{\lim_n} \|S^n\|^{1/n} \quad .$$

Without loss of generality it can be assumed that S = H is hyponormal. Let K be a contraction operator such that $H^* = KH$. Set $a_n = \|H^n\|$, $n = 1,2,\ldots$. Then

$$a_1^2 = \|H\|^2 = \|H^*H\| = \|KH^2\| \leq \|H^2\| = a_2 \ .$$

Similarly,

$$a_n^2 = \|H^n\|^2 = \|(H^*)^n H^n\| = \|(H^*)^{n-1} K H^{n+1}\| \leq \|(H^*)^{n-1}\| \ \|H^{n+1}\| = a_{n-1} a_{n+1} \ ,$$

for $n = 2,3,\ldots$. From Lemma 1.2, we obtain

$$a_1^n = \|H\|^n \leq \|H^n\| = a_n, \ n = 1,2,\ldots \quad .$$

Therefore, $\|H^n\| = \|H\|^n$, $n = 1,2,\ldots$. It follows that $r_{sp}(H) = \|H\|$. The proposition is proved.

The following corollary of Proposition 1.2 gives a growth condition on the resolvent of a seminormal operator:

Corollary 1.1. Let S be a seminormal operator. Assume λ_0 is a complex number such that $\lambda_0 \notin \sigma(S)$. Then

$$\|(S-\lambda_0)^{-1}\| = \frac{1}{\text{dist}(\lambda_0,\sigma(S))} ,$$

where dist $(\lambda_0,\sigma(S))$ denotes the distance from λ_0 to $\sigma(S)$.

Proof. From Proposition 1.2

$$\|(S-\lambda_0)^{-1}\| = \max\{|\lambda| : \lambda \in \sigma((S-\lambda_0)^{-1})\} .$$

This latter quantity equals $[\min\{|\lambda-\lambda_0| : \lambda \in \sigma(S)\}]^{-1}$. This ends the proof.

In the remainder of this section we discuss the usual manner for splitting off a maximal normal part from a seminormal operator. The part remaining used to be referred to as "completely non-normal" or "abnormal" part. More recently, the adjective "pure" has been used to describe this part. We will use this newer terminology.

The seminormal operator $S \in L(\mathcal{H})$ is called <u>pure</u> in case, the only subspace reducing S on which S is a normal operator is the zero subspace.

If H is a hyponormal operator on $\mathcal{H}$ and $\mathcal{M} \subset \mathcal{H}$ is an invariant subspace of H, then the restriction $H|_{\mathcal{M}}$ of H to $\mathcal{M}$ is also hyponormal. In fact, assume $H\mathcal{M} \subset \mathcal{M}$ and let P be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. If K is a contraction operator satisfying $H^* = KH$, then $(H|_{\mathcal{M}})^* = PH^*|_{\mathcal{M}} = PKPH|_{\mathcal{M}}$. The operator PKP (considered here as acting on $\mathcal{M}$) is a contraction and the remark follows. The same remark cannot be made for cohyponormal operators. For example, the restriction of a unitary operator to an invariant subspace can be an isometric non-normal (hence non-cohyponormal) operator. Nevertheless, as the reader may trivially check, the restriction of a seminormal operator to a reducing subspace is seminormal. Finally, we note the following useful remark.

4°. Let H be a hyponormal operator on $\mathcal{H}$. Assume $\mathcal{M} \subset \mathcal{H}$ is an invariant subspace for H such that the restriction operator $H|_{\mathcal{M}}$ is normal. Then $\mathcal{M}$ reduces the operator H.

The analogue of 4° formulated for cohyponormal operators is not true. The verification of 4° is left as an exercise.

The following theorem often enables the study of seminormal operators to be reduced to the case of a pure seminormal operator.

Theorem 1.1. Let S be seminormal operator on $\mathcal{H}$. Denote by $\mathcal{M}_0(S)$ the smallest subspace of $\mathcal{H}$ reducing the operator S containing the range of $D = [S^*,S]$. Set $\mathcal{M}_1(S) = \mathcal{M}_0(S)^\perp$. Relative to the decomposition $\mathcal{H} = \mathcal{M}_0(S) \oplus \mathcal{M}_1(S)$, the operator $S = S_0 \oplus S_1$, where S_0 is a pure seminormal operator and S_1 is normal.

Proof. Without loss of generality, it can be assumed that S=H is hyponormal. In this case $D = [H^*,H]$ is non-negative semidefinite. It suffices to show that any reducing subspace for H on which the restriction of H is normal must be orthogonal to $\mathcal{M}_0(H)$. Let $\mathcal{M}$ be such a subspace and let $f \in \mathcal{M}$. Then

$$0 = \|Hf\|^2 - \|H^*f\|^2 = (Df,f) = \|D^{1/2}f\|^2 \quad .$$

Thus $Df = 0$, or in other words, f is orthogonal to R(D). This ends the proof.

2. Examples.

In this section we assemble some common examples of semi-normal operators.

Note first, on a finite dimensional Hilbert space every seminormal operator S is normal. Indeed, trace$[S^*,S] = 0$. Since all the eigenvalues of $[S^*,S]$ have the same sign, this forces $[S^*,S] = 0$.

Let N be a normal operator on a Hilbert space $\mathcal{K}$. Assume $\mathcal{H}$ is an invariant subspace of N. The restriction $A = N|\mathcal{H}$ is called a subnormal operator on $\mathcal{H}$. Let P denote the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$ and $P' = I-P$ be the complementary projection, then

$$[A^*,A] = (PN^*N-NPN^*)|_{\mathcal{H}} = PNP'N^*|_{\mathcal{H}} .$$

The last operator is clearly positive and this establishes the well known result that every subnormal operator is hyponormal. It is interesting to note that subnormal operators used to be called hyponormal operators. It is not trivial to construct a hyponormal operator which is not subnormal (see, Halmos [1,p.107]).

As specific examples of subnormal operators we mention the following.

1°. Let ℓ_2 and ℓ_2^+ be the Hilbert spaces of square summable complex sequences of the form $\{c_n\}_{n=-\infty}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$, respectively. Let $\mathcal{H}$ be an auxiliary Hilbert space and write

$$\ell_2(\mathcal{H}) = \ell_2 \otimes \mathcal{H}, \quad \ell_2^+(\mathcal{H}) = \ell_2^+ \otimes \mathcal{H} .$$

In other words $\ell_2(\mathcal{H})$ $(\ell_2^+(\mathcal{H}))$ consists of the $\mathcal{H}$-valued square summable sequences of the form $\{f_n\}_{n=-\infty}^{\infty}$ $(\{f_n\}_{n=0}^{\infty})$. The operator U on $\ell_2(\mathcal{H})$ defined by

$$U\{f_n\}_{n=-\infty}^{\infty} = \{f_{n-1}\}_{n=-\infty}^{\infty}$$

is referred to as a bilateral shift with multiplicity equal to the dimension of $\mathcal{H}$. The operator U is clearly unitary. The subspace

$\ell_2^+(\mathcal{H})$ (viewed naturally as a subspace of $\ell_2(\mathcal{H})$) is an invariant subspace for U. The subnormal operator $U_+ = U|\ell_2^+(\mathcal{H})$ is called a <u>vector-valued unilateral shift</u>. The range of the self-commutator $[U_+^*, U_+]$ consists of elements $\{f_n\}_{n=0}^{\infty} \in \ell_2^+(\mathcal{H})$, for which $f_n = 0$, when $n > 0$. It follows immediately, from Theorem 1.1 of Section 1, that the operator U_+ is pure.

2°. Let μ be a non-negative finite Borel measure in the complex plane having compact support K. The closure of the polynomials (in the variable z) in $L^2(d\mu)$ is denoted by $H^2(d\mu)$. The closure of the rational functions with poles off K is denoted by $R^2(d\mu)$. There holds

$$H^2(d\mu) \subset R^2(d\mu) \subset L^2(d\mu) ,$$

where we must admit the possible equality between any of these spaces. The subspaces $H^2(d\mu)$ and $R^2(d\mu)$ are obviously invariant under the normal operator M_z defined on $L^2(d\mu)$ by

$$M_z f(z) = zf(z) .$$

The subnormal operators $M_z|_{H^2}$ and $M_z|_{R^2}$ have received considerable attention. A result of Bram [1] establishes that every subnormal operator with a cyclic vector is unitarily equivalent to an operator of the form $M_z|_{H^2(d\mu)}$, for some measure μ.

3°. In this example we use the notations $\ell_2(\mathcal{H})$ and $\ell_2^+(\mathcal{H})$ introduced in Example 1°. Let $\{A_n\}_{n=-\infty}^{\infty}$ be a sequence of operators on $\mathcal{H}$ satisfying

$$\|A_n\| \leq M, \; n = 0, \pm 1, \pm 2, \; \ldots ,$$

where $M > 0$ is a constant. The operator B on $\ell_2(\mathcal{H})$ defined by

$$B\{f_n\}_{n=-\infty}^{\infty} = \{A_n f_{n-1}\}_{n=-\infty}^{\infty}$$

is called an <u>operator valued bilateral weighted shift</u>. Similarly, if $\{A_n\}_{n=0}^{\infty}$ is a sequence of operators on $\mathcal{H}$ satisfying

$$\|A_n\| \leq M, \ n = 0,1,2,\ldots,$$

then we define the <u>operator valued unilateral weighted shift</u> A on $\ell_2^+(\mathcal{H})$ by

$$A\{f_n\}_{n=0}^{\infty} = \{A_n f_{n-1}\}_{n=0}^{\infty} \ (f_{-1} = 0).$$

The bilateral weighted shift B is hyponormal if and only if $A_n^* A_n \geq A_{n-1}^* A_{n-1}$, for all n. Similarly, the unilateral weighted shift A is hyponormal if and only if $A_n^* A_n \geq A_{n-1}^* A_{n-1}$, for all $n \geq 1$.

The question of purity of the weighted shift operators seems complicated. Let us consider a more specific example. Let V and D be non-negative self-adjoint operators on $\mathcal{H}$. We will assume the range of V is dense. Set $A_n = \sqrt{V+D}$, $n \geq 0$, and $A_n = \sqrt{V}$, $n < 0$. The bilateral shift B in this case has the matrix representation

$$B = \begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \sqrt{V} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & \sqrt{V} & \boxed{0} & 0 & \cdots \\ \cdots & 0 & 0 & \sqrt{V+D} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \sqrt{V+D} & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

and the self-commutator of B has the form

$$[B^*,B] = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & \boxed{D} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}.$$