INTRODUCTION TO

TOPOLOGY

Bert Mendelson

ASSOCIATE PROFESSOR OF MATHEMATICS, SMITH COLLEGE
MASSACHUSETTS, U.S.A.

With a Preface to the British Edition by

E. C. Zeeman, M.A., Ph.D.

FELLOW OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE

Blackie & Son Limited

LONDON GLASGOW

Introduction to Topology

Preface

BY DR. E. C. ZEEMAN

Topology and algebra are the two main pillars upon which modern mathematics is built. The student who has grasped the elementary principles of these two subjects discovers a simplicity and coherence running through the whole of mathematics. It is the elementary principles that are important rather than the technical details, and the idea of this book is to start from nothing and gently introduce the reader to the elementary principles of topology.

Topology is the study of continuity, and underlies every subject in mathematics and science that uses continuity. In particular topology underlies calculus, and logically ought to be taught before calculus. Often the schoolboy who cannot understand calculus when he first meets it is a better mathematician than the one who swallows it. The limiting process of calculus is a sophisticated technical process, inclined to impart a flavour of cheating to the boy who has previously been happy with the precision of arithmetic, algebra and geometry. This is disastrous because haziness of understanding produces fear of mathematics. Such a boy ought to be given this book. If he is mature enough to start calculus then he is mature enough to understand topology. By grasping the topological principles underlying calculus he will recapture that feeling of precision.

Ten years ago topology was taught, if at all, only in the third year at English universities, but today in recognition of the logical place of the subject, most universities introduce the elementary principles in the first year. This book covers exactly that first year material. In ten years' time it may well be taught in schools.

Compared with previous introductions to the subject, this book has two outstanding features. First, it is written from a geometrical point of view: there are plenty of diagrams and the reader is encouraged to draw his own, and to think geometrically. This is important because today the habit of geometrical thinking influences even the most abstract mathematical subjects. Also in higher dimensions, the most geometrical subject in mathematics is no longer geometry but algebraic topology, which is the sequel to this book.

Secondly, this book contains all the essential material and no more; there is room to relax, and there are sufficient exercises and examples to make it an ideal introduction at any level, be it school, university or scientific laboratory.

Preface

The first chapter consists of the usual discussion of set theory. The concept of a diagram consisting of sets and functions has been introduced at the same time. The concepts of equivalence relation and countability have been reserved for mention later, in Chapters IV and V respectively, where they make a natural appearance in connection with other topics.

The second chapter is a discussion of metric spaces, where the topological terms *open set*, *neighbourhood*, etc., have been carefully introduced. Particular attention is paid to various distance functions which may be defined on Euclidean n-space and which lead to the ordinary topology.

In the third chapter, topological space is introduced as a generalization of metric space. A great deal of attention has been paid to alternative procedures for the creation of a topological space, using neighbourhoods, etc., in the hope that this seemingly trivial, but subtle, point may be clarified. Since topological space is a generalization of metric space, it is hoped that the reader will observe the similarity, or perhaps redundancy, in the presentation of these two topics.

Chapters IV and V are devoted to a discussion of the two most important topological properties, connectedness and compactness. As applications the reader is introduced to a little algebraic topology. In Chapter IV to explain simple connectedness the concepts of homotopy and the fundamental group are described, except that the group structure is omitted because the reader is not presumed to know any group theory. Chapter V is concluded with a discussion of two-dimensional closed surfaces.

May, 1963 E. C. ZEEMAN

British Edition first published in 1963

BLACKIE & SON LIMITED
5 FITZHARDINGE STREET
PORTMAN SQUARE
LONDON · W.1
17 STANHOPE STREET
GLASGOW · C.4
BLACKIE & SON
(INDIA) LIMITED
103-5 FORT STREET
BOMBAY

*Printed and bound by Blackie & Son Ltd.
at the Villafield Press, Glasgow*

Contents

I Theory of Sets

II Metric Spaces

Theory of Sets

I

1 Introduction

As in any other branch of mathematics today, topology consists of the study of collections of objects that possess a mathematical structure. This remark should not be construed as an attempt to define *mathematics*, especially since the phrase "mathematical structure" is itself a vague term. We may, however, illustrate this point by two important examples.

The set of *positive integers* or *natural numbers* is a collection of objects N on which there is defined a function s, called the *successor function*, satisfying the conditions:

1. For each object x in N, there is one and only one object y in N such that $y = s(x)$;
2. Given objects x and y in S such that $s(x) = s(y)$, then $x = y$;
3. There is one and only one object in N, denoted by 1,

which is not the successor of an object in N, i.e., $1 \neq s(x)$ for each x in N;

4. Given a collection T of objects in N such that 1 is in T and for each x in T, $s(x)$ is also in T, then $T = N$.

The four conditions enumerated above are referred to as *Peano's axioms for the natural numbers.* The fourth condition is called *the principle of mathematical induction.* One defines addition of natural numbers in such a manner that $s(x) = x + 1$, for each x in N, which explains the use of the word "successor" for the function s. What is significant at the moment is the conception of the natural numbers as constituting a certain collection of objects N with an additional mathematical structure, namely the function s.

We shall use the system of real numbers as a second example of the fact that the type of entity that one studies in mathematics is a collection of objects with a certain mathematical structure. This explanation will require several preliminary definitions.

A *commutative field* is a collection of objects F and two functions that associate to each pair a, b of objects from F an element $a + b$ of F, called their sum, and an element $a\ b$ of F, called their product, respectively, satisfying the conditions:

1. For each a, b in F, $a + b = b + a$;
2. For each a, b, c in F, $a + (b + c) = (a + b) + c$;
3. There is a unique object in F, denoted by 0, such that $a + 0 = 0 + a = a$ for each a in F;
4. For each a in F, there is a unique object a' in F such that $a + a' = a' + a = 0$;
5. For each a, b in F, $a \cdot b = b \cdot a$;
6. For each a, b, c in F, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$;
7. There is a unique object in F, different from 0, denoted by 1, such that $a \cdot 1 = 1 \cdot a = a$ for each a in F;
8. For each a in F, if a is different from 0, there is a unique object a^* in F such that $a \cdot a^* = a^* \cdot a = 1$;
9. For each a, b, c in F, $a \cdot (b + c) = a \cdot b + a \cdot c$.

A commutative field F is thus a set of objects and an addition and multiplication that satisfies rules analogous to the rules of addition and multiplication of real numbers.

A field is called *linearly ordered* if it has additional structure, namely a relation "$<$" which satisfies the properties of "less than" as used in the real number system. Precisely, a field F is called a *linearly ordered* field if there is a relation "$<$" among certain ordered pairs of objects of F satisfying the conditions:

1. For each pair of objects x, y in F, one and only one of the three statements, $x < y$, $x = y$, $y < x$, is true;
2. For each object z in F, $x < y$ implies $x + z < y + z$;
3. For each object z in F such that $0 < z$, $x < y$ implies $x \cdot z < y \cdot z$.

Let T be a subcollection of objects from a linearly ordered field F. An object b in F is called an *upper bound* of T if for each x in T, either $x < b$ or $x = b$. An object a in F is called a *least upper bound* of T, if a is an upper bound of T and if $a < b$, where b is any other upper bound of T.

As a final definition before describing the system of real numbers, a linearly ordered field F is called *complete* if every non-empty subcollection T of F that has an upper bound also has a least upper bound. We can now state that the real number system is a collection R of objects together with operations of addition and multiplication and a relation "$<$" such that the collection R, together with this structure, is a complete, linearly ordered, commutative field.

[The use of the definite article, "*the* real number system is . . . ," should not be construed as asserting that there is only one real number system, but it is implicitly asserted that the conditions imposed on the collection R are categorical; that is, that any two instances of the real number system are indistinguishable, apart from the names or notation used to denote the objects.]

Thus we see that some of the better-known mathematical objects of study are describable as collections of objects together with certain specified structures. We shall describe a *topological space* in the same terms, although an appreciation of the utility of this concept can only come later. A topological space is a collection of objects (these objects usually being referred to as points), and a structure that endows this collection of points with some coherence, in the sense that we may speak of nearby points or points that in some sense are close together. This structure can be prescribed by means of a collection of subcollections of points called *open sets*. As we shall see, the major use of the concept of a topological space is that it provides us with an exact, yet exceedingly general setting for discussions that involve the concept of continuity.

By now the point should have been made that topology, as well as other branches of mathematics, is concerned with the study of collections of objects with certain prescribed structures. We therefore begin the study of topology by first studying collections of objects, or, as we shall call them, *sets*.

2 Sets and Subsets

We shall assume that the terms "object," "set," and the relation "is a member of" are familiar concepts. We shall be concerned with using these concepts in a manner that is in agreement with the ordinary usage of these terms.

If an object A belongs to a set S we shall write $A \in S$ (read, "A in S"). If an object A does not belong to a set S we shall write $A \notin S$ (read, "A not in S"). If $A_1, \ldots, A_n$ are objects, the set consisting of precisely these objects will be written

$$\{A_1, \ldots, A_n\}.$$

For purposes of logical precision it is often necessary to distinguish the set $\{A\}$, consisting of precisely one object A from the object A itself. Thus

$$A \in \{A\}$$

is a true statement, whereas

$$A = \{A\}$$

is a false statement. It is also necessary that there be a set that has no members, the so-called *null* or *empty* set. The symbol for this set is $\emptyset$ (a letter in the Swedish alphabet).

Let A and B be sets. If for each object $x \in A$, it is true that $x \in B$, we say that A is a *subset* of B. In this event, we shall also say that *A is contained in B*, which we write

$$A \subset B,$$

or that B *contains* A, which we write

$$B \supset A.$$

In accordance with the definition of subset, a set A is always a subset of itself. It is also true that the empty set is a subset of A. These two subsets, A and $\emptyset$, of A are called *improper* subsets, whereas any other subset is called a *proper* subset.

There are certain subsets of the real numbers that are frequently considered in calculus. For each pair of real numbers a, b with $a < b$, the set of all real numbers x such that $a \leqq x \leqq b$ is called the *closed interval* from a to b and is denoted by $[a, b]$. Similarly, the set of all real numbers x such that $a < x < b$ is called the *open interval* from a to b and is denoted by (a, b). We thus have $(a, b) \subset [a, b] \subset R$, where R is the set of real numbers.

Two sets are identical if they have precisely the same members. Thus, if A and B are sets, $A = B$ if and only if* both $A \subset B$ and $B \subset A$. Frequent use is made of this fact in proving the equality of two sets.

Sets may themselves be objects belonging to other sets. For example, $\{\{1, 3, 5, 7\}, \{2, 4, 6\}\}$ is a set to which there belong two objects, these two objects being the set of odd positive integers less than 8 and the set of even positive integers less than 8. If A is any set, there is available as objects with which to constitute a new set the collection of subsets of A. In particular, for each set A, there is a set we denote by 2^A whose members are the subsets of A. Thus, for each set A, we have $B \in 2^A$ if and only if $B \subset A$.

Exercises

1. Determine whether each of the following statements is true or false:

(a) For each set A, $\emptyset \in A$.

(b) For each set A, $\emptyset \subset A$.

(c) For each set A, $A \subset A$.

(d) For each set A, $A \in \{A\}$.

(e) For each set A, $A \in 2^A$.

(f) For each set A, $A \subset 2^A$.

(g) For each set A, $\{A\} \subset 2^A$.

(h) $\emptyset \in \{\emptyset\}$.

(i) For each set A, $\emptyset \in 2^A$.

(j) For each set A, $\emptyset \subset 2^A$.

(k) There are no members of the set $\{\emptyset\}$.

(l) Let A and B be sets. If $A \subset B$, then $2^A \subset 2^B$.

* The compound statement "P if and only if Q," is the conjunction of the two statements "If P then Q" and, "If Q then P." A statement of the form "P if and only if Q" may also be phrased "If P then Q and conversely."

(m) There are two distinct objects that belong to the set $\{\emptyset, \{\emptyset\}\}$.

2. Let A, B, C be sets. Prove that if $A \subset B$ and $B \subset C$, then $A \subset C$.
3. Let $A_1, \ldots, A_n$ be sets. Prove that if $A_1 \subset A_2$, $A_2 \subset A_3, \ldots,$ $A_{n-1} \subset A_n$ and $A_n \subset A_1$, then $A_1 = A_2 = \ldots = A_n$.
4. Let A be a set to which there belong precisely n distinct objects. Prove that there are precisely 2^n distinct objects that belong to 2^A.

3 Set Operations: Union, Intersection, and Complement

If x is an object, A a set, and $x \in A$, we shall say that x is an *element, member,* or *point* of A. Let A and B be sets. The *intersection* of the sets A and B is the set whose members are those objects x such that $x \in A$ and $x \in B$. The intersection of A and B is denoted by

$$A \cap B$$

(read, "A intersect B"). The *union* of the sets A and B is the set whose members are those objects x such that x belongs to at least one of the two sets A, B; that is, either $x \in A$ or $x \in B$.* The union of A and B is denoted by

$$A \cup B$$

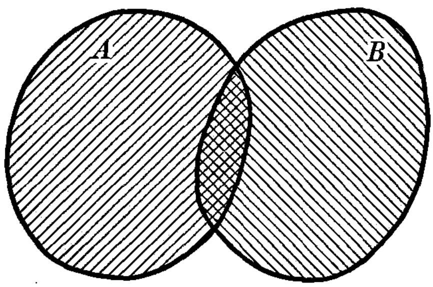
(read, "A union B").

The operations of set union and set intersection may be represented pictorially (by *Venn diagrams*). In Figure 1, let

* The logical connective "or" is used in mathematics (and also in logic) in the inclusive sense. Thus, a compound statement "P or Q" is true in each of the three cases: (1) P true, Q false; (2) P false, Q true; (3) P true, Q true, whereas "P or Q" is false only if both P and Q are false.

the elements of the set A be the points in the region shaded by lines running from the lower left-hand part of the page to the upper right-hand part of the page, and let the elements of the set B be the points in the region shaded by lines sloping in the opposite direction. Then the elements of $A \cup B$ are the points in a shaded region and the elements of $A \cap B$ are the points in a cross-hatched region.

Figure 1



Let $A \subset S$. The *complement* of A in S is the set of elements that belong to S but not to A. The complement of A in S is denoted by $C_S(A)$ or by $S - A$. The set S may be fixed throughout a given discussion, in which case the complement of A in S may simply be called the complement of A and denoted by $C(A)$. $C(A)$ is again a subset of S and one may take its complement. The complement of the complement of A is A; that is, $C(C(A)) = A$.

There are many formulas relating the set operations of intersection, union, and complementation. Frequent use is made of the following two formulas.

Theorem (DeMorgan's Laws) Let $A \subset S$, $B \subset S$. Then

(3.1) $$C(A \cup B) = C(A) \cap C(B),$$

(3.2) $$C(A \cap B) = C(A) \cup C(B).$$

Proof. Suppose $x \in C(A \cup B)$. Then $x \in S$ and $x \notin A \cup B$.

Thus, $x \notin A$ and $x \notin B$, or $x \in C(A)$ and $x \in C(B)$. Therefore $x \in C(A) \cap C(B)$ and, consequently,

$$C(A \cup B) \subset C(A) \cap C(B).$$

Conversely, suppose $x \in C(A) \cap C(B)$. Then $x \in S$ and $x \in C(A)$ and $x \in C(B)$. Thus, $x \notin A$ and $x \notin B$, and therefore $x \notin A \cup B$. It follows that $x \in C(A \cup B)$ and, consequently,

$$C(A) \cap C(B) \subset C(A \cup B).$$

We have thus shown that

$$C(A) \cap C(B) = C(A \cup B).$$

One may prove Formula 3.2 in much the same manner as 3.1 was proved. A shorter proof is obtained if we apply 3.1 to the two subsets $C(A)$ and $C(B)$ of S, thus

$$C(C(A) \cup C(B)) = C(C(A)) \cap C(C(B)) = A \cap B.$$

Taking complements again, we have

$$C(A) \cup C(B) = C(C(C(A) \cup C(B))) = C(A \cap B).$$

Exercises

1. Let $A \subset S$, $B \subset S$. Prove the following:

(a) $\emptyset = C(S)$.	(f) $A \cup A = A$.
(b) $S = C(\emptyset)$.	(g) $A \cup S = S$.
(c) $A \cap C(A) = \emptyset$.	(h) $A \cap S = A$.
(d) $A \cup C(A) = S$.	(i) $A \cup \emptyset = A$.
(e) $A \cap A = A$.	(j) $A \cap \emptyset = \emptyset$.

(k) $A \subset B$ if and only if $A \cup B = B$.
(l) $A \subset B$ if and only if $A \cap B = A$.
(m) $A \cup B = B$ if and only if $A \cap B = A$.
(n) $A \subset C(B)$ if and only if $A \cap B = \emptyset$.
(o) $C(A) \subset B$ if and only if $A \cup B = S$.
(p) $A \subset B$ if and only if $C(B) \subset C(A)$.
(q) $A \subset C(B)$ if and only if $B \subset C(A)$.