
FOUNDATIONS
OF

ELECTRO-
DYNAMICS

S.R. DE GROOT AND L.G. SUTTORP

Foundations of Electrodynamics

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Preface

Electrodynamics may be said to consist of two parts, at different levels: microscopic and macroscopic theory. The first contains the laws that govern the interaction of fields and point particles – often grouped into stable sets such as atoms and molecules – and the second those that describe the interaction of fields and continuous media. The two theories are linked together, since the phenomena at the macroscopic level may be looked upon as being the result of the interplay of many particles. Therefore one should be able to obtain the electromagnetic laws for continuous media from those for point particles. Such a derivation, together with a discussion of the microscopic starting points, forms the subject of this monograph.

The programme will be carried out in the framework of both classical and quantum theory. The classical theory is given in the non-relativistic approximation and then in covariant formulation. In the latter various topics will receive special attention: among these figure the covariant description of composite particles, the obtention of statistical averages in a relativistically invariant way and a discussion of the energy-momentum tensor for continuous media. The quantum-mechanical theory will be formulated in such a fashion that the analogy with classical theory can be exploited as far as possible. This is achieved by representing the physical quantities by ordinary functions rather than by operators. Again the non-relativistic approximation will be studied first. Subsequently magnetic effects are discussed in a 'semi-relativistic' theory, which goes one step beyond the non-relativistic treatment. The completely covariant extension of quantum theory will be confined to the discussion of the motion of single particles with and without spin in slowly varying external fields. The covariant generalization to statistical assemblies of particles moving in each other's fields would require quantization of the electromagnetic field together with its sources: this forms the subject of quantum electrodynamics not dealt with here.

The subject matter of the various chapters is, roughly spoken, of two kinds.

Part is meant especially to serve as textbook material for graduate students who take courses in electromagnetic theory. By reading the first two chapters they will get acquainted with the way in which the macroscopic laws of electrodynamics are obtained from a microscopic basis, albeit in the framework of classical, non-relativistic theory. In the relativistic part the third chapter may be useful as an exposé of the covariant equations for fields and particles with the inclusion of the effects of radiation damping, while the final results of the fourth and fifth chapters give an idea of the way in which the non-relativistic laws may be generalized. Similarly the results of chapters VI and VII show the consequences of the use of quantum mechanics. The special formulation of quantum mechanics in terms of Weyl transforms and Wigner functions can be studied independently from the appendix of chapter VI.

More advanced students will be interested in the covariant formulation of the equations of motion for composite particles in chapter IV, relativistic statistics as discussed in chapter V, the covariant quantum-mechanical equations of motion for particles with spin 0 and $\frac{1}{2}$ in chapter VIII, and in the semi-relativistic treatment of magnetic effects given in chapters IX and X.

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S.R. de G.

L.G.S.

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Table of contents

Preface v

Table of contents vii

PART A NON-RELATIVISTIC CLASSICAL ELECTRODYNAMICS

Ch. I *Particles: their fields and motion* 3

§1 Introduction 3

§2 The microscopic field equations 3

§3 The equation of motion for a point particle 5

§4 The equations for the fields due to composite particles 6

The atomic series expansion. Multipole moments. The field equations.

§5 The momentum and energy equations for composite particles 11

The equation of motion. The energy equation.

§6 The inner angular momentum equation for composite particles 16

Problems 19

Ch. II *Statistical description of fields and matter* 21

§1 Macroscopic laws 21

§2 Average quantities 22

§3 The Maxwell equations 24

§4 Applications 30

The polarizations up to dipole moments. The polarizations up to quadrupole moments. Examples: metals, insulators, plasmas, fluids, electrolytes.

§5 The momentum and energy equations 37

Introduction. The mass conservation law. The momentum balance. The energy balance. The short range terms in the momentum and energy laws. The momentum and energy equations for fluids. Mixtures, in particular plasmas. Crystalline solids. Galilean invariance.

§6 The angular momentum equations 60

The inner angular momentum balance. Fluid systems. Plasmas. Crystalline solids. Galilean invariance.

§7 The laws of thermodynamics 67

The first law. The second law for fluids. The second law for plasmas. The second law for crystalline solids. The entropy balance equation.

§8 Helmholtz and Kelvin forces 95

Fluids. Crystalline solids.

Appendix I *On the depolarizing tensor* 106Appendix II *The Hamiltonian for a system of composite particles in an external field* 110Appendix III *Deformations and free energy* 114

Problems 117

PART B COVARIANT FORMULATION OF CLASSICAL ELECTRODYNAMICS

Ch. III *Charged point particles* 127

§1 Introduction 127

§2 The field equations 127

Covariant formulation. The solutions of the field equations. Expansion of the retarded and advanced potentials and fields into powers of c^{-1} . The Liénard-Wiechert potentials and fields. The self-field of a charged particle.

§3 The equation of motion 146

A single particle in a field. A set of particles in a field.

Appendix *On an energy-momentum tensor with 'local' character* 157

Problems 164

Ch. IV *Composite particles* 168

§1 Introduction 168

§2 The field equations 168

The atomic series expansion. Multipole moments. The field equations. Explicit expressions for the polarization tensor. The non-relativistic and semi-relativistic limits.

§3 The equations of motion of a composite particle in a field 180

Introduction. Definition of a covariant centre of energy. Charged dipole particles. Charged particles with magnetic dipole moment proportional to their inner angular momentum. Composite particles in an arbitrarily varying electromagnetic field. A set of composite particles in a field.

Appendix I *Some properties of the tensor Ω* 209

Appendix II *The connexion between the covariant and the atomic multipole moments* 211

Appendix III *On equations of motion with explicit radiation damping* 217

Appendix IV *The minus field of a charged dipole particle* 221

Appendix V *Semi-relativistic equations of motion for a composite particle* 225

The semi-relativistic approximation. The momentum and energy equations. The angular momentum equation.

Problems 240

Ch. V *Covariant statistics: the laws for material media* 245

§1 Introduction 245

§2 Covariant statistical mechanics 246

Covariant distribution functions. Definition of macroscopic quantities. Synchronous, retarded and advanced distribution functions.

§3 The Maxwell equations 255

Derivation of the macroscopic field equations. Explicit forms of the macroscopic current vector and polarization tensor.

§4 The conservation of energy-momentum 262

The conservation of rest mass. Energy-momentum conservation for a fluid system of neutral atoms. Energy-momentum conservation for a neutral plasma. The macroscopic energy-momentum tensor. The ponderomotive force density.

§5 The conservation of angular momentum 279

The balance equation of inner angular momentum. The ponderomotive torque density.

§6 Relativistic thermodynamics of polarized fluids and plasmas 287

The first law. The second law. The free energy for systems with linear constitutive relations. The energy-momentum tensor for a polarized fluid at local equilibrium. Induced dipole and permanent dipole substances. The generalized Helmholtz force density.

§7 On the uniqueness of the energy-momentum tensor 300

Problems 308

PART C NON-RELATIVISTIC QUANTUM-MECHANICAL ELECTRODYNAMICS

Ch. VI *The Weyl formulation of the microscopic laws* 317

§1 Introduction 317

§2 The field equations and the equations of motion of a set of charged point particles 317

§3 The Weyl transformation and the Wigner function 319
The Weyl transformation. The Wigner function.

§4 The Weyl transforms of the field equations 328

§5 The Weyl transform of the equation of motion 331

§6 The equations for the fields of composite particles 333

§7 The momentum and energy equations for composite particles 336

The equation of motion. The energy equation.

§8 The inner angular momentum equation for composite particles 339

Appendix *Properties of the Weyl transformation and the Wigner function* 341

A reformulation of quantum mechanics. The Weyl transformation. The Wigner function. Generalization to particles with spin.

Problems 365

Ch. VII *Quantum statistical description of material media* 369

§1 Introduction 369

§2 The Wigner function in statistical mechanics 369

§3 Reduced Wigner functions 371

§4 The Maxwell equations 373

§5 The momentum and energy equations 375

Introduction. The mass conservation law. The momentum balance. The energy balance. The short range terms. The correlation contributions. Substances with short range correlations.

§6 The angular momentum equations 384

§7 The laws of thermodynamics 385

The first law. The second law.

Appendix I *The Wigner function in statistical mechanics* 391

Definition. Properties. Reduced Wigner functions. The reduced Wigner function for a perfect gas.

Appendix II *The Hamilton operator for a system of composite particles in an external field* 399

Appendix III *Deformations and free energy in quantum theory* 403

Problems 405

PART D: COVARIANT AND SEMI-RELATIVISTIC QUANTUM-MECHANICAL ELECTRODYNAMICS

Ch. VIII *Dirac and Klein-Gordon particles in external fields* 409

§1 Introduction 409

§2 The free Dirac particle 410

Invariances of the Dirac equation. Covariance requirements on position and spin. Transformation of the Hamiltonian to even form; the position and spin operators.

§3 The Dirac particle in a field 425

Invariance properties. Covariance requirements on the position and spin operators for a particle in a field. Weyl transforms for particles with spin. Transformation of the Hamilton operator. Covariant position and spin operators. Equations of motion and of spin.

§4 The free Klein-Gordon particle 439

The Klein-Gordon equation and its transformation properties. Feshbach and Villars's formulation. Covariance requirements on the position operator. Transformation to even form of the Hamilton operator; the position operator.

§5 The Klein-Gordon particle in a field 446

Invariance properties. Covariance requirements on the position operator. The transformed Hamilton operator. The position operator and the equation of motion.

Appendix *On covariance properties of physical quantities for the Dirac and Klein-Gordon particles* 452

The Dirac equation in covariant notation. Local covariance and Klein's theorem. Covariance requirements on the position and spin operator of the free Dirac particle. Three mutually excluding requirements on the position operator for the free Dirac particle. On the uniqueness of the position operator of the free Klein-Gordon particle.

Problems 462

Ch. IX *Semi-relativistic description of particles with spin* 466

§1 Introduction 466

§2 The Hamilton operator up to order c^{-2} for a system of Dirac and Klein-Gordon particles in an external field 466

§3 The field equations and the equations of motion for a set of spin particles 475

§4 The semi-relativistic approximation 480

§5 The equations for the fields due to composite particles 481

§6 The laws of motion for composite particles 484

The equation of motion. The energy equation. The angular momentum equation.

Ch. X *Semi-relativistic quantum statistics of spin media* 493

§1 Introduction 493

§2 The Wigner function in statistics; particles with spin 493

§3 The Maxwell equations 496

§4 The momentum and energy equations 497

Conservation of rest mass. The momentum balance. The energy balance.
The angular momentum balance.

§5 The laws of thermodynamics 509

The first law. The second law.

§6 Applications 515

Appendix I *The Hamilton operator for a set of composite particles with spin* 520Appendix II *Change of free energy under deformations for a spin particle system* 524

Author index 527

Subject index 529

PART A

Non-relativistic classical electrodynamics

CHAPTER I

Particles: their fields and motion

1 Introduction

The aim of this chapter is to obtain a description of the electromagnetic behaviour of composite particles in the framework of classical, non-relativistic theory. Such composite particles, like atoms, molecules or ions are supposed to consist of charged point particles: the electrons and nuclei. The equations which govern their motion and describe their fields will be derived from the corresponding basic equations valid for charged point particles without structure. The latter microscopic equations are the Maxwell-Lorentz field equations and the Newton equation with the Lorentz force inserted. A series expansion in terms of multipoles leads then to the field equations and the momentum, energy and angular momentum equations for the composite particles.

2 The microscopic field equations

The electric and magnetic fields $\mathbf{e}(\mathbf{R}, t)$ and $\mathbf{b}(\mathbf{R}, t)$ at the point with coordinates $\mathbf{R}$ and at time t , generated by a collection of point particles $i = 1, 2, \dots$ with charges e_i , positions $\mathbf{R}_i(t)$ and velocities $\dot{\mathbf{R}}_i(t)$, satisfy the Maxwell-Lorentz field equations (in the rationalized Gauss system¹)

$$\begin{aligned} \nabla \cdot \mathbf{e} &= \sum_i e_i \delta(\mathbf{R}_i - \mathbf{R}), \\ -\partial_0 \mathbf{e} + \nabla \wedge \mathbf{b} &= c^{-1} \sum_i e_i \mathbf{R}_i \delta(\mathbf{R}_i - \mathbf{R}), \\ \nabla \cdot \mathbf{b} &= 0, \\ \partial_0 \mathbf{b} + \nabla \wedge \mathbf{e} &= 0, \end{aligned} \tag{1}$$

¹ In the Giorgi system different numerical coefficients appear: the factors c^{-1} (both explicitly and in ∂_0) are absent, while in the first two equations $\mathbf{e}$ and $\mathbf{b}$ are replaced by $\epsilon_0 \mathbf{e}$ and $\mu_0^{-1} \mathbf{b}$ respectively.

where $\dot{\nabla}$ and ∂_0 are differentiations with respect to $\mathbf{R}$ and ct (with c the speed of light) and the dot and the symbol $\wedge$ scalar and vector products of vectors. The sources contain the three-dimensional delta functions of $\mathbf{R}_i - \mathbf{R}$.

In non-relativistic theory one is interested in solutions of these equations up to order c^{-1} . To find them it is convenient to introduce potentials. From the third equation it follows that

$$\mathbf{b} = \dot{\nabla} \wedge \mathbf{a} \quad (2)$$

with the vector potential $\mathbf{a}(\mathbf{R}, t)$. Then with the fourth equation one has

$$\mathbf{e} = -\nabla\varphi - \partial_0 \mathbf{a}, \quad (3)$$

where $\varphi(\mathbf{R}, t)$ is the scalar potential. Insertion of these expressions into the first two equations of (1) gives, if one omits terms in c^{-2} ,

$$\begin{aligned} \Delta\varphi + \partial_0 \nabla \cdot \mathbf{a} &= -\sum_i e_i \delta(\mathbf{R}_i - \mathbf{R}), \\ \Delta \mathbf{a} - \nabla(\nabla \cdot \mathbf{a} + \partial_0 \varphi) &= -c^{-1} \sum_i e_i \dot{\mathbf{R}}_i \delta(\mathbf{R}_i - \mathbf{R}), \end{aligned} \quad (4)$$

where $\Delta = \nabla \cdot \nabla$ is the Laplace operator. The potentials are not fixed in a unique way by the relations (2) and (3). The same electromagnetic fields are described by potentials $\mathbf{a}'$ and φ' which are related to the original potentials $\mathbf{a}$ and φ by a gauge transformation

$$\begin{aligned} \varphi' &= \varphi - \partial_0 \psi, \\ \mathbf{a}' &= \mathbf{a} + \nabla \psi \end{aligned} \quad (5)$$

with an arbitrary function ψ . This property is utilized to choose the potentials in such a way that they satisfy

$$\partial_0 \varphi + \nabla \cdot \mathbf{a} = 0, \quad (6)$$

the Lorentz condition. The reason for imposing this condition is that then the equations (4) become two uncoupled Poisson equations for φ and $\mathbf{a}$:

$$\begin{aligned} \Delta\varphi &= -\sum_i e_i \delta(\mathbf{R}_i - \mathbf{R}), \\ \Delta \mathbf{a} &= -c^{-1} \sum_i e_i \dot{\mathbf{R}}_i \delta(\mathbf{R}_i - \mathbf{R}), \end{aligned} \quad (7)$$

where again a term of order c^{-2} has been dismissed. The solutions follow from the property for the delta function:

$$\Delta \frac{1}{|\mathbf{r}|} = -4\pi\delta(\mathbf{r}). \quad (8)$$

Using this property, one finds from (7) the non-relativistic potentials in the Lorentz gauge:

$$\begin{aligned}\varphi &= \sum_i \frac{e_i}{4\pi|\mathbf{R}_i - \mathbf{R}|}, \\ a &= c^{-1} \sum_i \frac{e_i \mathbf{R}_i}{4\pi|\mathbf{R}_i - \mathbf{R}|},\end{aligned}\quad (9)$$

so that the non-relativistic fields (2) and (3) are

$$\begin{aligned}e &= \sum_i e_i, & e_i &= -\nabla \frac{e_i}{4\pi|\mathbf{R}_i - \mathbf{R}|}, \\ b &= \sum_i b_i, & b_i &= c^{-1} \nabla \wedge \frac{e_i \dot{\mathbf{R}}_i}{4\pi|\mathbf{R}_i - \mathbf{R}|}.\end{aligned}\quad (10)$$

These formulae show that the non-relativistic electric field is of order c^0 (a term in c^{-1} does not appear), while the non-relativistic magnetic field is of order c^{-1} (no term in c^0 arises). From the first line of (10) it follows that e is irrotational. This is in agreement with the fourth field equation in (1), since $\partial_0 b$ is of order c^{-2} and hence has to be neglected in non-relativistic theory. So strictly spoken one should write in a non-relativistic theory the truncated equation $\nabla \wedge e = 0$ instead of the fourth field equation.

3 The equation of motion for a point particle

The equation of motion for a particle with charge e , mass m , position $\mathbf{R}_1(t)$, velocity $\dot{\mathbf{R}}_1(t)$ and acceleration $\ddot{\mathbf{R}}_1(t)$ in an external electromagnetic field ($\mathbf{E}_e, \mathbf{B}_e$) is:

$$m\ddot{\mathbf{R}}_1 = e\{\mathbf{E}_e(\mathbf{R}_1, t) + c^{-1}\dot{\mathbf{R}}_1 \wedge \mathbf{B}_e(\mathbf{R}_1, t)\}, \quad (11)$$

where at the right-hand side the Lorentz force appears. The equation of motion of one particle of a set labelled by the index $i = 1, 2, \dots, N$ reads

$$m_i \ddot{\mathbf{R}}_i = e_i \{e_i(\mathbf{R}_i, t) + c^{-1} \dot{\mathbf{R}}_i \wedge b_i(\mathbf{R}_i, t)\}, \quad (12)$$

where the total electric and magnetic fields are the sums of the external fields and the fields (10) generated by the other particles:

$$\begin{aligned}e_i(\mathbf{R}_i, t) &= \sum_{j(\neq i)} e_j(\mathbf{R}_i, t) + \mathbf{E}_e(\mathbf{R}_i, t), \\ b_i(\mathbf{R}_i, t) &= \sum_{j(\neq i)} b_j(\mathbf{R}_i, t) + \mathbf{B}_e(\mathbf{R}_i, t).\end{aligned}\quad (13)$$

Since in the equation of motion (12) the magnetic field is accompanied by a factor c^{-1} , one needs there as fields

$$\begin{aligned} e_i(\mathbf{R}_i, t) &= - \sum_{j(i \neq j)} \nabla_i \frac{e_j}{4\pi|\mathbf{R}_i - \mathbf{R}_j|} + E_e(\mathbf{R}_i, t), \\ b_i(\mathbf{R}_i, t) &= B_e(\mathbf{R}_i, t), \end{aligned} \quad (14)$$

instead of the complete expressions (13) with (10).

The equations of motion (12) with (14) may be written in Hamiltonian form

$$\frac{\partial H}{\partial \mathbf{P}_i} = \dot{\mathbf{R}}_i, \quad \frac{\partial H}{\partial \mathbf{R}_i} = -\dot{\mathbf{P}}_i \quad (15)$$

with the Hamiltonian

$$H = \sum_i \frac{\mathbf{P}_i^2}{2m_i} + \sum_{i, j(i \neq j)} \frac{e_i e_j}{8\pi|\mathbf{R}_i - \mathbf{R}_j|} + \sum_i e_i \left\{ \varphi_e(\mathbf{R}_i, t) - c^{-1} \frac{\mathbf{P}_i \cdot \mathbf{A}_e(\mathbf{R}_i, t)}{m_i} \right\}, \quad (16)$$

with φ_e and $\mathbf{A}_e$ potentials for the external fields. Indeed insertion of (16) into (15) leads to (12) with (14).

4 The equations for the fields due to composite particles

a. The atomic series expansion

Charged point particles (electrons and nuclei) are often grouped into stable sets, like atoms, molecules or ions. (For convenience we shall sometimes refer to such composite particles simply as 'atoms'.) The starting point for the derivation of the equations for the fields due to such atoms is the set of microscopic field equations (1). It will be convenient in the present case to replace the numbering i of the point particles by a numbering k of the stable groups and i of their constituent particles. The position vector $\mathbf{R}_i$, written as $\mathbf{R}_{ki}$ now, can be split into two parts:

$$\mathbf{R}_{ki} = \mathbf{R}_k + \mathbf{r}_{ki}. \quad (17)$$

Here $\mathbf{R}_k$ is the position of some privileged point of the stable group k (e.g. the nucleus of an atom or the centre of mass, etc.), while the $\mathbf{r}_{ki}$ ($i = 1, 2, \dots$) are the internal coordinates, which specify the positions of the constituent particles ki with respect to that of the privileged point of the stable group k .

The case will now be studied in which the solutions e and b of the field equations can be considered as converging series expansions in $|\mathbf{r}_{ki}|/|\mathbf{R}_k - \mathbf{R}|$.