Lecture Notes in Physics Series — Vol. 40

PHASE SPACE PICTURE OF QUANTUM MECHANICS

Group Theoretical Approach

Lecture Notes in Physics Series — Vol. 40

PHASE SPACE PICTURE OF QUANTUM MECHANICS

Group Theoretical Approach

Y. S. Kim
Department of Physics and Astronomy
University of Maryland
College Park, Maryland 20742, USA

M. E. Noz
Department of Radiology
New York University
New York 10016, USA

World Scientific
Singapore • New Jersey • London • Hong Kong

Published by

World Scientific Publishing Co. Pte. Ltd.
P O Box 128, Farrer Road, Singapore 9128
USA office: 687 Hartwell Street, Teaneck, NJ 07666
UK office: 73 Lynton Mead, Totteridge, London N20 8DH

PHASE-SPACE PICTURE OF QUANTUM MECHANICS

ISBN 981-02-0360 8
981-02 0361 6 (pbk)

Printed in Singapore by Utopia Press.

PREFACE

Quantum mechanics can take different forms. The Schrödinger picture of quantum mechanics is very useful in atomic and nuclear physics. The Heisenberg picture is the basic language for the covariant formulation of quantum field theory. Is there then any need for a new picture of quantum mechanics? This depends on whether there are branches of physics where the Schrödinger or Heisenberg picture is less than fully effective.

Quantum optics and relativistic bound-state problems are relatively new fields. In quantum optics, we deal with creation and annihilation of photons and linear superposition of multiphoton states. It is possible to construct the mathematics of harmonic oscillators in the Schrödinger picture to describe the photon's states. However, the mathematics becomes complicated when we attempt to describe generalized coherent states often called the squeezed states. Is there a language simpler than the Schrödinger picture?

Quantum field theory accommodates both the uncertainty principle and special relativity. However, it is less than fully effective in describing bound-state problems or localized probability distributions. It is possible to construct models of relativistic hadrons consisting of quarks starting from the Schrödinger picture of quantum mechanics. The question then is whether it is possible to formulate the uncertainty relations in a covariant manner (Dirac 1927).

The phase-space picture of quantum mechanics provides the answer to these questions. Starting from the Schrödinger wave function, it is possible to construct a distribution function, often called the Wigner function, in phase space in terms of the c-number position and momentum variables. In this picture, it is possible to perform canonical transformations as in the case of classical mechanics. This will bring us a deeper understanding of the uncertainty principle.

This phase-space picture of quantum mechanics is not new. The earliest application of the Wigner phase-space distribution function was made in quantum corrections to thermodynamics in 1932 (Wigner 1932a). Since then, the Wigner function has been discussed in many branches of physics including statistical mechanics, nuclear physics, atomic and molecular physics, and foundations of physics. However, it is difficult to see the advantage of using the Wigner function over the existing method in those traditional branches of physics.

In this book, we discuss applications of the Wigner function in quantum optics and the relativistic quark model which are relatively new subjects in physics and which still need a basic scientific language. From the mathematical point of view, the Wigner function for the ground-state harmonic oscillator is the basic language for these new branches of physics. However, its symmetry properties constitute the most interesting aspect of this new scientific language.

Indeed, the symmetry property of the Wigner function in phase space is that of the Lorentz group. The Lorentz group is known to be a difficult subject to mathematicians, because it is a non-compact group. To physicists, group theory is a difficult subject when its representations have no physical applications. However, the situation is quite the opposite when the representation can extract physical implications.

In this book, we discuss the physical consequences of the symmetries of the Wigner function in phase space. This book is written for those scientists and students who wish to study the basic principles of the phase-space picture of quantum mechanics and physical applications of the Wigner distribution functions. This book will also serve a useful purpose for those who simply wish to study the physical applications of the Lorentz group.

We are indebted to Professor Eugene P. Wigner for encouraging us to formulate a group theoretical approach to the phase-space picture of quantum mechanics. Professor Wigner suggested the use of the light-cone coordinate system for the covariant formulation of the Wigner function. Indeed, Chapter 10 of this book is based on Professor Wigner's ideas. He suggested the possibility that the work of Inonu and Wigner (1953) on group contractions be extended to study the space-time geometry of relativistic particles (Kim and Wigner 1987a and 1990a). He also suggested the use of the concept of entropy when the measurement process is less than complete in a relativistic system (Kim and Wigner 1990c).

While this book was being written, we received helpful comments and suggestions from many of our colleagues, including K. Cho, D. Han, C. H. Kim, M. Kruger, P. McGrath, H. S. Pilloff, L. Rana, Y. H. Shih, J. Soln, C. Van Hine, and W. W. Zachary.

September 1990
YSK and MEN

INTRODUCTION

The concept of phase space arises naturally from the Hamiltonian formulation of classical mechanics, and plays an important role in the transition from classical physics to quantum theory. However, in quantum mechanics, the position and momentum variables cannot be measured simultaneously. In the Schrödinger picture, the wave function is written as a function of either the position or the momentum variable, but not of both. For this reason, in quantum mechanics, the density matrix (Von Neumann 1927 and 1955) replaces phase space as a device for describing the density of states. It therefore appears that phase space is not a useful concept in quantum mechanics. We disagree. The role of phase space in quantum mechanics has not yet been fully explored.

Starting from the density matrix, is it possible to develop an algorithm of quantum mechanics based on phase space? This question has been raised repeatedly since the publication in 1932 of Wigner's paper on the quantum correction for thermodynamic equilibrium (Wigner 1932a). Since it is not possible to measure simultaneously position and momentum without error, it is meaningless to define a point in phase space. However, this does not prevent us from defining an area element in phase space whose size is not smaller than Planck's constant. Since the measurement problem is stated in terms of the least possible value of the product of the uncertainties in the position and momentum, it is of interest to see how the uncertainty product can be stated in phase space.

The basic advantage of this phase-space picture of quantum mechanics is that it is possible to perform canonical transformations, just as in classical mechanics. The purpose of this book is to study the physical consequences derivable from canonical transformations in quantum mechanics. Using these transformations, we can compare quantum mechanics with classical physics in terms of many illustrative examples. In addition, the phase-space picture of quantum mechanics is becoming a new scientific language for modern optics which is a rapidly expanding field. Furthermore, the Lorentz transformation in a given direction of boost is a canonical transformation in the light-cone coordinate system. This allows us to state the uncertainty relation in a Lorentz-invariant manner.

There are still many questions concerning the uncertainty relations for which answers are not well known. For instance, in the Schrödinger picture, the free-particle wave packet becomes widespread, and the uncertainty product increases as time progresses or regresses. Is it possible to state the uncertainty relation in terms of the quantity which remains constant? Can phase space provide an answer to this question? The answer to this question is YES. In the phase space-picture, the uncertainty is define in terms of the area which the Wigner distribution function occupies. The spread of a wave packet is an area-preserving canonical transformation in the phase-space picture of quantum mechanics.

Quantum optics is a rapidly expanding subject, and it is increasingly clear that coherent and squeezed states of light will play a major role in a new understanding of the uncertainty principle, and will provide innovations in high-technology industrial applications. These optical states are minimum-uncertainty states, and transformations among these state are therefore canonical transformations. Indeed, the phase-space picture of quantum mechanics is the natural language for these relatively new quantum states.

Most physicists these days learn classical mechanics from Goldstein's textbook (Goldstein 1980). However, Goldstein's book does not emphasize the importance of linear canonical transformations, which are discussed in more advanced books (Arnold 1978, Abraham and Marsden 1978, Guilemin and Sternberg 1984). In this book, we shall discuss the group of linear canonical transformations in phase space which is the inhomogeneous symplectic group (Han *et al.* 1988). For a single pair of canonically conjugate variables, the group is the inhomogeneous symplectic group ISp(2), and it is ISp(4) for two pairs of conjugate variables.

If we do not take into account translations in phase space, the symmetry groups become those of homogeneous symplectic transformations. The groups Sp(2) and Sp(4) are locally isomorphic to the (2 + 1)-dimensional and (3 + 2)-dimensional Lorentz groups. Thus the study of the symmetries in phase space requires the study of Lorentz transformations.

The Lorentz transformation is one of the most fundamental transformations in physics, and this subject can be formulated in terms of the inhomogeneous Lorentz group (Wigner 1939). Since this group governs the fundamental space-time symmetries of elementary particles, there are many papers and books on this subject (Kim and Noz 1986). In this book, we treat Lorentz transformations as canonical transformations.

One of the persisting question in modern physics is whether the uncertainty relations can be Lorentz-transformed. Does Planck's constant remain invariant under Lorentz transformations? Is localization of the probability distribution a Lorentz-invariant concept? It is very difficult to answer these questions in the Heisenberg or Schrödinger picture of quantum mechanics. The basic limitation of these pictures is that they do not tell us how the uncertainty relations appear to observers in different Lorentz frames. The question of whether quantum mechanics can be made consistent with special relativity has been and still is the central issue of modern physics.

We shall address this question within the framework of the phase-space picture of quantum mechanics. It is interesting to note that the Lorentz boost in a given direction is a canonical transformation in phase space using the light-cone variables. This allows us to state the uncertainty relations in a Lorentz-invariant manner. Feynman's parton picture (Feynman 1969) and the nucleon form factors are discussed as illustrative examples.

In the first two Chapters, we discuss the forms of classical mechanics and quantum mechanics useful for the formulation of the Wigner phase-space picture of quantum mechanics, which is discussed in detail in Chapters 3 and 4. Chapters 5 and 6 are for the applications of the Wigner function to coherent and squeezed states of light. It is seen in these chapters that the study of the Wigner function requires the knowledge of the Lorentz group.

In Chapters 7 and 8, we present a detailed discussion of the physical representations of the inhomogeneous Lorentz group or the Poincaré group which governs the fundamental space-time symmetries of relativistic particles. By constructing the representation based on harmonic oscillators, we study the phase-space picture of relativistic extended particles. Chapters 9 contains a detailed discussion of experimental observation of Lorentz-squeezed hadrons. Finally, in Chapter 10, we discuss some fundamental issues in space-time symmetries of relativistic system, including the unification of space-time symmetries of massive and massless particles and the entropy increase due to the incompleteness in measurements.

Since we are combining the Wigner function with group theory, we have reprinted in the Appendix Wigner's 1932 paper on the Wigner function as well as his 1939 paper on the representations of the inhomogeneous Lorentz group. The study of phase space requires a knowledge of harmonic oscillators. P.A.M. Dirac was interested in constructing representations of the Lorentz group based on four dimensional harmonic oscillators. We have therefore included Dirac's 1945 paper on the Lorentz group and his 1963 paper on the de Sitter group.

There are many other interesting subjects which can be studied within the framework of the phase-space picture of quantum mechanics but are not discussed in this book. However, there are now a number of review articles (Wigner 1971, O'Connell 1983, Carruthers and Zachariasen 1983, Hillery *et al.* 1984, Balazs and Jennings 1984, Littlejohn 1986) containing applications of the Wigner phase-space distribution function to various branches of modern physics. The scope of this book is limited to the simplest form of the Wigner function with maximum symmetry applicable to the branches of physics in which the phase-space picture is definitely superior to other forms of quantum mechanics.

PHASE SPACE PICTURE OF QUANTUM MECHANICS

Group Theoretical Approach

Contents

Preface v

Introduction vii

1 PHASE SPACE IN CLASSICAL MECHANICS 1
1.1 Hamiltonian Form of Classical Mechanics 1
1.2 Trajectories in Phase Space 4
1.3 Canonical Transformations 6
1.4 Coupled Harmonic Oscillators 9
1.5 Group of Linear Canonical Transformations in Four-Dimensional Phase Space 12
1.6 Poisson Brackets 14
1.7 Distributions in Phase Space 15

2 FORMS OF QUANTUM MECHANICS 19
2.1 Schrödinger and Heisenberg Pictures 19
2.2 Interaction Representation 22
2.3 Density-Matrix Formulation of Quantum Mechanics 25
2.4 Mixed States 27
2.5 Density Matrix and Ensemble Average 30
2.6 Time Dependence of the Density Matrix 32

3 WIGNER PHASE-SPACE DISTRIBUTION FUNCTIONS 37
3.1 Basic Properties of the Wigner Phase-Space Distribution Function . 38
3.2 Time Dependence of the Wigner Function 40
3.3 Wave Packet Spreads 42
3.4 Harmonic Oscillators 45
3.5 Density Matrix 47
3.6 Measurable Quantities 49
3.7 Early and Recent Applications 53

4 LINEAR CANONICAL TRANSFORMATIONS IN QUANTUM

MECHANICS 57
4.1 Canonical Transformations in Two-Dimensional Phase Space . . . 57
4.2 Linear Canonical Transformations in Quantum Mechanics 60
4.3 Wave Packet Spreads in Terms of Canonical Transformations 63
4.4 Harmonic Oscillators 64
4.5 (2 + 1)-Dimensional Lorentz Group 66
4.6 Canonical Transformations in Four-Dimensional Phase Space . . . 69
4.7 The Schrödinger Picture of Two-Mode Canonical Transformations . 72
4.8 (3 + 2)-Dimensional de Sitter Group 73

5 COHERENT AND SQUEEZED STATES 77
5.1 Phase-Number Uncertainty Relation 78
5.2 Baker-Campbell-Hausdorff Relation 80
5.3 Coherent States of Light 83
5.4 Symmetry Groups of Coherent States 86
5.5 Squeezed States 89
5.6 Two-Mode Squeezed States 93
5.7 Density Matrix through Two-Mode Squeezed States 96

6 PHASE-SPACE PICTURE OF COHERENT AND SQUEEZED STATES 99
6.1 Invariant Subgroups 100
6.2 Coherent States 102
6.3 Single-Mode Squeezed States 105
6.4 Squeezed Vacuum 107
6.5 Expectation Values in terms of Vacuum Expectation Values 110
6.6 Overlapping Distribution Functions 113
6.7 Thomas Effect 115
6.8 Two-Mode Squeezed States 117
6.9 Contraction of Phase Space 120

7 LORENTZ TRANSFORMATIONS 123
7.1 Group of Lorentz Transformations 124
7.2 Little Groups of the Lorentz Group 125
7.3 Massless Particles 127
7.4 Decomposition of Lorentz Transformations 129
7.5 Analytic Continuation to the Little Groups for Massless and Imaginary-Mass particles 131
7.6 Light-Cone Coordinate System 133
7.7 Localized Light Waves 135
7.8 Covariant Localization of Light Waves 138

7.9 Covariant Phase-Space Picture of Localized Light Waves 140
7.10 Uncertainty Relations for Light Waves and for Photons 142

8 COVARIANT HARMONIC OSCILLATORS 145
8.1 Theory of the Poincaré Group 146
8.2 Covariant Harmonic Oscillators 149
8.3 Irreducible Unitary Representations of the Poincaré Group 154
8.4 C-number Time-Energy Uncertainty Relation 155
8.5 Dirac's Form of Relativistic Theory of "Atom" 157
8.6 Lorentz Transformations of Harmonic Oscillator Wave functions . . . 160
8.7 Covariant Phase-Space Picture of Harmonic Oscillators 162

9 LORENTZ-SQUEEZED HADRONS 167
9.1 Quark Model 168
9.2 Hadronic Mass Spectra 171
9.3 Hadrons in the Relativistic Quark Model 174
9.4 Form Factors of Nucleons 178
9.5 Phase-Space Picture of Overlapping Wave Functions 180
9.6 Feynman's Parton Picture 184
9.7 Experimental Observation of the Parton Distribution 188

10 SPACE-TIME GEOMETRY OF EXTENDED PARTICLES 193
10.1 Two-Dimensional Euclidean Group and Cylindrical Group 195
10.2 Contractions of the Three-Dimensional Rotation Group 197
10.3 Three-Dimensional Geometry of the Little Groups 200
10.4 Little Groups in the Light-Cone Coordinate System 202
10.5 Cylindrical Group and Gauge Transformations 204
10.6 Little Groups for Relativistic Extended Particles 207
10.7 Lorentz Transformations and Hadronic Temperature 210
10.8 Decoherence and Entropy 213

A Reprinted Articles 217
A.1 E.P. Wigner, On the Quantum Correction for Thermodynamic Equilibrium 219
A.2 E.P. Wigner, On Unitary Representations of the Inhomogeneous Lorentz Group 233
A.3 P.A.M. Dirac, Unitary Representations of the Lorentz Group . . 293
A.4 P.A.M. Dirac, A Remarkable Representation of the 3 + 2 de Sitter Group 307

References 319
Index 329

Chapter 1

PHASE SPACE IN CLASSICAL MECHANICS

The concept of phase space originates from classical mechanics in its Hamiltonian formalism, in which a given dynamical system depends on a number of independent coordinate variables and the same number of conjugate momentum variables. The Cartesian space consisting of these $2n$ coordinate variables is called phase space (Goldstein 1980).

This phase-space formalism is the starting point for the modern approach to classical mechanics (Arnold 1978, Abraham and Marsden 1978), including nonlinear dynamics and chaos. Traditionally, the phase-space formalism of classical mechanics plays the role of a bridge between classical mechanics and quantum mechanics. In this Chapter, we shall study the properties of classical phase space which will be shared by the phase-space formulation of quantum mechanics.

Of particular interest are linear canonical transformations which correspond to unitary transformations in the Schrödinger picture of quantum mechanics. The mathematics for linear canonical transformations is that of the symplectic group which is relatively new in physics (Weyl 1946). The linear transformations of the n pairs of canonical variables is governed by the group $Sp(2n)$ (Gilmore 1974, Guillemin and Sternberg 1984). In this book, we will be primarily concerned with physical problems requiring one and two pairs of canonical variables. With this point in mind, we shall start this section with the Hamiltonian formulation of classical mechanics.

1.1 Hamiltonian Form of Classical Mechanics

Classical mechanics starts with Newton's second law stating that force is proportional to acceleration. There are several reformulations of this law such as the Lagrangian and Hamiltonian formalisms. The Lagrangian form is useful when we do not wish to consider constraint forces. It plays the key role in quantum field theory. It is also serves as the bridge between Newton's second law and the Hamiltonian formalism.

If there are n independent coordinates $q_1, q_2, \cdots, q_n$ in a given dynamical system, the Lagrangian is a function of these coordinates and their time derivatives, as well as the time variable:

$$L = L(q_1, q_2, \cdots, q_n; \dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n; t). \tag{1.1}$$

From this, the momentum variable conjugate to q is defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{1.2}$$

For each i, the equation of motion is

$$\dot{p}_i = \frac{\partial L}{\partial q_i}. \tag{1.3}$$

This is the Lagrangian form of the equation of motion.

The Hamiltonian is defined as

$$H = \sum_i \dot{q}_i p_i - L. \tag{1.4}$$

Then, from the Lagrangian equations of motion,

$$\delta H = \sum_i \left(\dot{q}_i \delta p_i - \dot{p}_i \delta q_i\right) + \frac{\partial H}{\partial t}\delta t. \tag{1.5}$$

Thus, for each i, we can write the Hamiltonian equation of motion as

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{1.6}$$

Now the Hamiltonian can be regarded as a function of $q_1, q_2, \cdots, q_n$ and $p_1, p_2, \cdots, p_n$.

As far as the time dependence is concerned, from the definition of the Hamiltonian given in Eq. 1.4,

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}. \tag{1.7}$$

The total derivative of the Hamiltonian is

$$\frac{dH}{dt} = \sum_i \left(\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i\right) + \frac{\partial H}{\partial t}. \tag{1.8}$$

As a consequence of the equations of motion, the quantity in parenthesis vanishes, and

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{1.9}$$

Thus, if the Lagrangian does not depend on time explicitly, the Hamiltonian is a constant of motion.

Let us consider some examples. The Hamiltonian for a free particle is naturally $H = p^2/2m$, and the Hamiltonian for n particles is

$$H = \sum_{i=1}^{n} p_i^2/2m_i, \tag{1.10}$$

which is the total energy. The equations of motion lead to $\dot{p}_i = 0$ for every i.

The Hamiltonian for a charged particle in an electromagnetic field generated by the vector potential $\mathbf{A}$ and the scalar potential ϕ is

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + e\phi, \tag{1.11}$$

where e is the charge of the particle. In this case, the momentum vector $\mathbf{p}$ is not the mass times the velocity, but the quantity $\frac{1}{m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)$ is the velocity. Since the magnetic field does not change the magnitude of velocity, the above Hamiltonian is the total energy of the system. This form of the Hamiltonian has been discussed extensively in standard textbooks on classical and quantum mechanics (Goldstein 1980, Schiff 1968). Since the potential can be gauge-transformed, the above Hamiltonian is not invariant under gauge transformations. However, the resulting equations of motion are invariant under gauge transformations.

For example, for a particle in a constant magnetic field along the z direction, $\mathbf{B} = \hat{e}_z B$, the vector potential can be written as

$$\mathbf{A} = \hat{e}_y x B, \quad \text{or} \quad \mathbf{A} = \frac{1}{2}\left(\hat{e}_y x - \hat{e}_x y\right) B. \tag{1.12}$$

The difference between the two potentials is $\frac{1}{2}\left(\hat{e}_y x + \hat{e}_x y\right) B$. This is the gradient of the scalar function $xyB/2$. These two potentials give the same set of equations of motion.

For the first choice of $\mathbf{A}$, the Hamiltonian can be written as

$$H = \frac{1}{2m}\left\{p_x^2 + (p_y - exB/c)^2\right\}. \tag{1.13}$$

Thus, according to the Hamiltonian equations of motion,

$$\dot{x} = \left(\frac{1}{m}\right) p_x, \quad \dot{y} = \frac{1}{m} p_y - \left(\frac{eB}{mc}\right) x,$$
$$\dot{p}_x = \left(\frac{eB}{mc}\right)\left(p_y - \frac{exB}{c}\right), \quad \dot{p}_y = 0. \tag{1.14}$$

From the first two equations,

$$p_x = m\dot{x}, \quad p_y = m\dot{y} + exB/c. \tag{1.15}$$

The substitution of these into Eq. 1.14 leads to the familiar set of Newton's equations for the circular orbit with the cyclotron frequency eB/mc. The Hamiltonian equations of motion will take a different form for the second vector potential in Eq. 1.12, but they will lead to the same set of Newton's equations.

The Hamiltonian for the one-dimensional harmonic oscillator is

$$H = \left(\frac{1}{2m}\right) p^2 + \left(\frac{K}{2}\right) x^2. \tag{1.16}$$

According to the equations of motion, $p = m\dot{x}$, and $Kx = -\dot{p}$. This result is well known. This form of the Hamiltonian plays the central role in modern optics and relativistic quantum mechanics, and will be discussed extensively in this book.

Let us consider the Galilei transformation of this system, where the coordinate is transformed as

$$x' = x + vt. \tag{1.17}$$

This means that the above harmonic oscillator is on a truck which moves with the velocity v. Then to the observer on the ground, the Hamiltonian will be

$$H = \left(\frac{1}{2m}\right) p'^2 + \left(\frac{K}{2}\right) (x' - vt)^2. \tag{1.18}$$

The equations of motion are

$$\dot{p}' = K(x' - vt), \quad \dot{x}' = p'/m. \tag{1.19}$$

This leads to the conclusion that the acceleration in the truck frame is the same as that on the ground. Furthermore, the Lagrangian can be written as

$$L = \left(\frac{m}{2}\right) \dot{x}'^2 - \left(\frac{K}{2}\right) (x' - vt)^2, \tag{1.20}$$

which leads to $p' = m\dot{x}' = m(\dot{x} + v)$. Thus

$$p' = p + mv. \tag{1.21}$$

Thus, in terms of x and p, the Hamiltonian becomes

$$H = \left(\frac{1}{2m}\right) (p + mv)^2 + \left(\frac{K}{2}\right) x^2. \tag{1.22}$$

This is consistent with what we expect from the Galilei transformation.

1.2 Trajectories in Phase Space

It is quite clear that, in the Hamiltonian formalism, the dynamical system of n degrees of freedom is described by n coordinate variables $q_1, q_2, \cdots, q_n$, and their conjugate momenta $p_1, p_2, \cdots, p_n$. It is then possible to consider a $2n$-dimensional space spanned by n coordinate and n momentum variables. This space is called phase space.

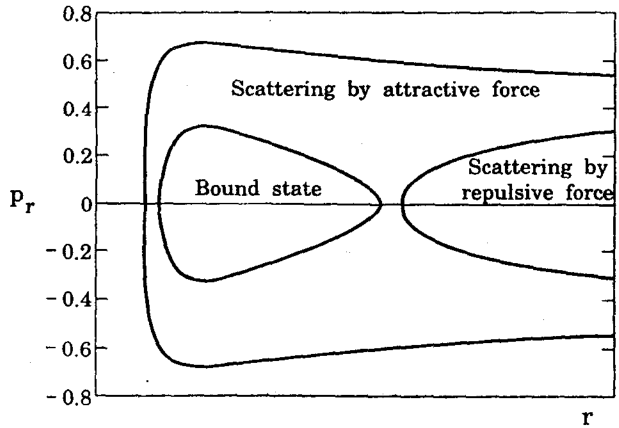

Figure 1.1: Phase-space description of the Kepler problem. If the total energy is negative, the orbit is closed. If the energy is positive, the orbit is open. The particle comes in with a negative momentum and goes out with a positive momentum. This figure gives an interesting description of the transition from negative to positive energy. If k is negative, the potential is repulsive, and the momentum takes its maximum value at an infinite distance.

If there is one degree of freedom, then a two-dimensional phase space consisting of x and p can completely determine the dynamical system. For one free particle with a given momentum p, the particle trajectory is a line parallel to the x axis with a fixed value of p. For the one-dimensional harmonic oscillator, the Hamiltonian is the total energy, and the trajectory is an ellipse in the phase space of x and p.

Let us next consider a simple pendulum whose Hamiltonian is (Goldstein 1980)

$$H = p_\theta^2/2I + mg\ell(1 - cos\theta). \tag{1.23}$$

Here again the Hamiltonian is the total energy. The angular momentum is given by p_θ. If the energy is very small, the small-angle approximation can be made, and thus the energy becomes

$$E = p_\theta^2/2I + (mg\ell/2)\theta^2. \tag{1.24}$$

Thus the trajectory in the phase space of θ and p_θ is an ellipse. If however, the energy is much larger than $mg\ell$, then the potential energy terms can be ignored, and the energy becomes that of a free particle.

It is possible to give a phase space description of the Kepler problem. The total energy can be written in terms of the radial momentum and the effective potential as (Goldstein 1980):

$$E = p_r^2/2m + \ell^2/2mr^2 - k/r, \tag{1.25}$$

where ℓ is the total angular momentum. If k is positive, the system can have a bound state. Then the trajectory in the phase space of r and p_r is closed for a negative value of E. If the energy is positive, the trajectory is open as is indicated