GIAN-CARLO ROTA, *Editor*
ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS
Volume 21

Section: Combinatorics
Gian-Carlo Rota, *Section Editor*

Graph Theory

W. T. Tutte

GIAN-CARLO ROTA, *Editor*
ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS
Volume 21

Section: Combinatorics
Gian-Carlo Rota, *Section Editor*

Graph Theory

W. T. Tutte
Faculty of Mathematics
University of Waterloo
Waterloo, Ontario, Canada

Foreword by
Crispin St. J. A. Nash-Williams
The University of Reading

1984

Addison-Wesley Publishing Company
Advanced Book Program
Menlo Park, California

Reading, Massachusetts • London • Amsterdam • Don Mills, Ontario • Sydney • Tokyo

Library of Congress Cataloging in Publication Data

Tutte, W. T.
Graph theory.

(Encyclopedia of mathematics and its applications;
v. 21)
Bibliography: p.
Includes index.
1. Graph theory. I. Title. II. Series.
QA166.T815 1984 511'.5 83-12210
ISBN 0-201-13520-5

American Mathematical Society (MOS) Subject Classification Scheme (1980): 05C

ABCDEFGHIJ-MA-8987654

Manufactured in the United States of America

Editor's Statement

A large body of mathematics consists of facts that can be presented and described much like any other natural phenomenon. These facts, at times explicitly brought out as theorems, at other times concealed within a proof, make up most of the applications of mathematics, and are the most likely to survive change of style and of interest.

This ENCYCLOPEDIA will attempt to present the factual body of all mathematics. Clarity of exposition, accessibility to the non-specialist, and a thorough bibliography are required of each author. Volumes will appear in no particular order, but will be organized into sections, each one comprising a recognizable branch of present-day mathematics. Numbers of volumes and sections will be reconsidered as times and needs change.

It is hoped that this enterprise will make mathematics more widely used where it is needed, and more accessible in fields in which it can be applied but where it has not yet penetrated because of insufficient information.

GIAN-CARLO ROTA

We are happy to inaugurate the Combinatorics Section of the Encyclopedia. Professor Tutte, one of the founders of graph theory, is presenting here the jewels of a subject rich in deep results, in a volume that will long remain definitive.

Foreword

It is both fitting and fortunate that the volume on graph theory in the *Encyclopedia of Mathematics and Its Applications* has an author whose contributions to graph theory are—in the opinion of many—unequaled. Indeed, the style and content of the book betray throughout the influence of Professor Tutte's own work and the distinctive flavor of his personal approach to the subject—a flavor familiar to those of use who have heard his beautifully constructed lectures on many occasions and are delighted to see so much of this exposition now recorded in permanent form for a wider audience.

The book deals with many of the central themes that one might expect to find in books on graph theory, such as Menger's Theorem and network flows, the Reconstruction Problem, the Matrix-Tree Theorem, the theory (largely created by Professor Tutte) of factors (or matchings) in graphs, chromatic polynomials, Brooks' Theorem, Grinberg's Theorem, planar graphs, and Kuratowski's Theorem. However, this is by no means "just another book on graph theory," since the treatment of all these topics is unified into a coherent whole by Professor Tutte's highly individual approach. Moreover, the more customary topics are leavened with some "pleasant surprises," such as the author's attractive theory of decomposition of graphs into 3-connected "3-blocks" (not found in other books except [5]), an interesting and remarkable approach to electrical networks, and—perhaps particularly—the classification theorem for closed surfaces. This theorem is usually considered as part of topology, but Professor Tutte shows us that it fits admirably into a work on graph theory, where indeed the essentially combinatorial nature of the arguments may be more at home.

From these remarks, it will be clear that the book has much to offer to any reader interested in graph theory. It draws together some important themes from the rapidly growing literature of the subject, and by no means duplicates any other expository writing at present available. It will also provide an excellent preparation for some slightly more specialized topics, including much of Professor Tutte's own work—for example, his extensive theory of planar enumeration and chromatic polynomials of maps, his theorem on Hamiltonian circuits in 4-connected planar graphs, the Five-Flow Conjecture (see Chapter IX, Sec. IX.4), and the interactions of matroids with graphs (see Chapter VIII, Sec. VIII.11). The treatment of minors provides background for (among many other things) the conjecture about well-quasi-ordering of graphs by the relationship of one graph being a minor of another (see the second half of Note 1 at the end of Chapter II). Current work of G. N. Robertson (a former student of Professor Tutte) and P. D. Seymour seems to promise exciting developments related to this problem. Some of these—and other—topics might provide ample material for a sequel volume, if Professor Tutte, or someone close to his thinking, is impelled to write one.

In the Introduction, Professor Tutte describes his early encounters with graph theory, particularly as a student at Cambridge. This cannot but provoke one's own early recollections, a favorite topic of conversation among graph-theorists being what first drew one's attention to graph theory. My own answer is "Nothing: I just invented it." In other words, when starting work as a research student, also at Cambridge, I felt that there ought to be a branch of mathematics dealing with this kind of thing, and, if there was not, I would create it. (References to binary relations in algebra courses might have helped to foster this idea.) It is a measure of the little known state of graph theory at that time that it took me some weeks to discover that I was not its first inventor and to hear of the one existing textbook on the subject, König's *Theorie der endlichen und unendlichen Graphen*, published eighteen years earlier in 1936. After König, I turned to the few research papers on graph theory then available, and I recall many hours in the library studying Professor Tutte's work on factors of graphs, liberally armed with colored pencils (especially red and bue ones, of course) to draw the diagrams needed to aid one's imagination. The reader is advised to equip himself similarly when he reaches Chapter VII, if indeed he has not already found this to be essential at a much earlier stage in the study of this or any other book on graph theory.

During the early part of my career, working in graph theory was a rarity and an idiosyncrasy. A graph-theorist could not expect to have others among his colleagues and might be hard pressed to find them in the same country: one simply did not expect to interact with other mathematicians except through published literature. Undergraduate and even graduate courses in the subject were virtually unknown. To some mathematicians, it even seemed questionable whether graph theory was worthwhile mathematics at all. Doubts seemed to center on its lack of elaborate techniques and its lack of unification and relevance, in the sense that it seemed to consist largely of solutions to isolated problems, not interacting closely with one another or with the rest of mathematics.

However, in fairness I must say that, for every mathematician who may have harbored such doubts, there were others who seemed sympathetic and interested. Not least among these, when I became a University lecturer, was my first Head of Department, Professor E. M. Wright (now Sir Edward Wright), whose own work has taken a distinctly graph-theoretic turn in more recent years.

In those early years, I would have thought a person almost demented if he had predicted the subsequent explosive growth in graph theory and other areas of combinatorial mathematics, which must be among the most remarkable experiences of those of us who have lived through it. Writers of futuristic novels do well to remember that truth is often stranger than fiction. It would have seemed absurd to predict that names like G. A. Dirac, F. Harary, and W. T. Tutte (then just meaningless cryptograms attached to research papers) would quite soon become those of some of the mathematicians whom I would know best personally, that graph theory would take me across the Atlantic to join a "Department of Combinatorics and Optimization," where for several years Professor Tutte and I would occupy adjacent offices, that there would be more combinatorial conferences than any one person could attend, and that the combinatorial literature would expand to its present size, including at least six journals (and perhaps more, depending on the method of counting) devoted entirely to combinatorics.

The first of these to emerge, the *Journal of Combinatorial Theory*, has for many years had Professor Tutte as its editor-in-chief (or in recent times, since proliferation of material forced it to subdivide, as editor-in-chief of half of it). The editor of this *Encyclopedia*, Professor G.-C. Rota, was one of the organizers responsible for getting the *Journal* started, and I was reliably informed by Blanche Descartes at that time that she had selected its title as an anagram of OUR FOE JIAN-CARLO ROTA BIMONTHLY.

Despite all this increased recognition of combinatorics (including graph theory) as part of mathematics, traces of the controversy about its value still persist. In the introduction to [1], L. Lovász rightly responds by pointing to the increasing body of techniques and unifying theory that the subject is acquiring. Nevertheless, some further remarks may be relevant. As another former colleague of mine, J. Sheehan, has suggested [4], there may still be important parts of combinatorics that do not fit well into any such unifying framework. There is surely *some* satisfaction in solving an obvious, natural and tough problem, even if the solution seems, at least for the present, to stand on its own. If (as sometimes seems to be suggested) a theorem can be significant only by virtue of its helpfulness in proving or illuminating other theorems, then one must go on to ask why those other theorems are significant, and a *reductio ad absurdum* is eventually reached. To prove that a branch of mathematics is interesting, must one necessarily demonstrate that it is exactly like other branches of mathematics, or might part of the appeal of mathematics lie in its diversity?

On the other hand, it is unquestionable that interplay between ideas from different sources, and elaborate techniques successfully applied, are *among* the features that make much of mathematics fascinating. Moreover, mathematics does often display a tendency to unify itself and to build up a body of technique. Therefore one may well guess (despite my caution against predicting the future) that graph theory, as it matures, will continue to develop its own characteristic techniques and that many of its results will become increasingly unified, both among themselves and with the rest of mathematics. The present book may be expected to play a considerable part in placing graph theory on a sound theoretical and technical footing.

I have written elsewhere what little I can about some aspects of the present state of graph theory [3] and Professor Tutte's immense influence on it [2]. This, and an early publication deadline, must be my excuse for a comparatively short Foreword consisting largely of random reminiscences and reflections that may interest nobody, with the possible exception of myself. Perhaps some future historian of mathematics may glean a crumb or two of something or other from them; but I will detain the reader no longer from the rich harvest in store.

C. St. J. A. Nash-Williams

REFERENCES

[1] L. Lovász, *Combinatorial Problems and Exercises*, North-Holland Publishing Company, Amsterdam, 1979.

[2] C. St. J. A. Nash-Williams, "A note on some of Professor Tutte's mathematical work," in *Graph Theory and Related Topics*, Proceedings of the conference held in honor of Professor W. T. Tutte on the occasion of his sixtieth birthday, University of Waterloo, July 5–9, 1977 (edited by J. A. Bondy and U. S. R. Murty), Academic Press, New York, 1979, pp. xxv–xxviii.

[3] C. St. J. A. Nash-Williams, "A glance at graph theory—Part I" and "A glance at graph theory—Part II," *Bulletin of the London Mathematical Society*, **14** (1982), 177–212 and 294–328.

[4] J. Sheehan, review of *Graph Theory: An Introductory Course* by B. Bollobás, *Bulletin of the London Mathematical Society*, **12** (1980), 388–390.

[5] W. T. Tutte, *Connectivity in Graphs*, University of Toronto Press, Toronto, 1966.

Introduction

The author encountered graph theory in high school, in the early thirties, while reading Rouse Ball's book *Mathematical Essays and Recreations*. He then learned of Euler paths (Sec. VI.3), map-colorings (dualized in Sec. IX.3), factors of graphs (Sec. VII.6), and Tait colorings (Sec. IX.5) [5].

As an undergraduate at Cambridge he joined with R. L. Brooks, C. A. B. Smith, and A. H. Stone in the study of their hobby-problem of dissecting a square into unequal squares [3]. This soon called for much graph theory. It was linked, through a "Smith diagram," with the study of 3-connected planar graphs (Sec. XI.7), and with Kirchhoff's Laws for electrical circuits (Sec. VI.5). It was linked through rotor theory (Sec. VI, Notes) with graph symmetry (Sec. I.2). It was linked through the tree-number (Sec. II.2) with the theory of graph functions satisfying simple recursion formulae (Sec. IX.1).

All this is explained in the Commentaries of [4]. That is one reason why I do not discuss squared rectangles and the analogous triangulated triangles in the present work. Another is that I visualize the book as a work on pure graph theory, making no appeal either to point-set topology or to elementary geometry.

I became acquainted with some graph-theoretical literature at Cambridge. I read Sainte-Laguë's description of the proof of Petersen's Theorem (Sec. VII.6). I found the classical papers of Hassler Whitney, published in 1931–3, and the famous book of Dénes König, the first textbook devoted entirely to graph theory. I was there at Cambridge at the time of the births of Smith's Theorem (Sec. IX.5) [7] and Brooks' Theorem (Sec. IX.3) [2]. Stone's discovery of flexagons came a little later.

Having meditated upon these things for 45 years I now present some of them in the present work. It is an attempt at the reference book I would have liked to have in 1936–40. In electrical theory it is important to know whether you have a connection between the two terminals, and what happens when you remove a wire. Chapter I deals graph-theoretically with these matters. Chapter II deals with the effect of contracting an edge, or shall we say making a short circuit. The theory of 3-connection is discussed in Chapter IV, and the halfway stage of 2-connection in Chapter III. Chapter V, on reconstruction, is less directly related to squared squares and rectangles. I came to it by way of reconstruction formulae for some of the above-mentioned recursive graph functions [11].

Chapter VI concerns digraphs and a generalized theory of Kirchhoff's Laws. It arose out of a study of triangulated triangles by the four undergraduates. We were sometimes reproached for basing our mathematical theory on physical laws. We protested, of course, that for us Kirchhoff's Laws were axioms of a purely mathematical system, but we were glad to be able to emphasize this by introducing generalized Laws, describing a kind of electricity that never was on land or sea.

Chapter VII derives from Sainte-Laguë's paper, with some gaps filled and some extensions made. Chapter VIII is about cycles and coboundaries, generalizations of Kirchhoff flows. It attempts to describe some parts of graph theory algebraically, and most of it derives from my doctoral thesis of 1948 [9].

Chapter VIII is about the recursive graph functions. It derives from a paper of 1947 [8]. It discusses the dichromatic polynomial, the dichromate, the chromatic polynomial, and the flow-polynomial, all of which can be referred to the theory of map-colorings and to the dual theory of vertex-colorings.

So far there is one important omission, that of a theory of planarity. The graphs of interest in connection with squared rectangles and triangulated triangles are all planar, so Chapter X prepares for the introduction of planarity by giving a general theory of maps on surfaces. But this is to be a purely graph-theoretical work, and so the maps of Chapter X are structures defined by purely combinatorial axioms. Surfaces are defined as classes of maps. The discussion is an adaptation of the classical theory of H. R. Brahana [1]. Planar maps can now be defined as maps of Euler characteristic 2.

Chapter XI gives a theory of planarity. It gives duality theorems for the tree-number and the dichromate, and it gives a combinatorial version of Jordan's Theorem. It goes on to some tests for the planarity or nonplanarity of a given graph, MacLane's and Kuratowski's among them. This part derives from my paper "How to draw a graph," of 1964, but it skips the actual drawing, that being a matter of elementary geometry rather than graph theory.

I take this opportunity to express my indebtedness to Brooks, Smith, and Stone, without whose missionary zeal I might now be writing on some other subject.

REFERENCES

[1] Brahana, H. R., Systems of circuits on two-dimensional manifolds, *Ann. Math.* (**2**) 23 (1921), 144–168.

[2] Brooks, R. L., On colouring the nodes of a network, *Proc. Cambridge Phil. Soc.* **37** (1941), 194–197.

[3] Brooks, R. L., C. A. B. Smith, A. H. Stone, and W. T. Tutte, The dissection of rectangles into squares, *Duke Math. J.* **7** (1940), 312–340.

[4] McCarthy, D., and R. G. Stanton (eds.), *Selected papers of W. T. Tutte*, The Charles Babbage Research Centre, St. Pierre, Manitoba, Canada (1979).

[5] Rouse Ball, W. W., *Mathematical recreations and essays*. The 12th edition (U. of Toronto Press, 1974) is co-authored by H. S. M. Coxeter.

[6] Sainte-Laguë, M. A., Les réseaux (ou graphes). *Memorial des Sciences Mathématiques*, Fasc. XVIII, Paris 1926.

[7] Tutte, W. T., On Hamiltonian circuits, *J. London Math. Soc.* **21** (1946), 98–101.

[8] __________, A ring in graph theory, *Proc. Cambridge Phil. Soc.* **43** (1947), 26–40.

[9] __________, An algebraic theory of graphs, Thesis, Cambridge 1948.

[10] __________, How to draw a graph, *Proc. London Math. Soc.* **52** (1963), 743–767.

[11] __________, On dichromatic polynomials, *J. Comb. Theory* **2** (1967), 301–320.

Contents

Chapter I

Graphs and Subgraphs

I.1. DEFINITIONS

In this section we give a formal definition of a graph and introduce some of the basic terminology of graph theory. We give examples of graphs and some sample theorems.

A *graph* G is defined by a set $V(G)$ of elements called *vertices*, a set $E(G)$ of elements called *edges*, and a relation of *incidence*, which associates with each edge either one or two vertices called its *ends*.

The present work is concerned only with *finite* graphs, those in which the sets $V(G)$ and $E(G)$ are both finite. Much interesting work has been done on the other graphs, the *infinite* ones. But even the theory of finite graphs is too big to be adequately covered in one volume. Let us therefore make the rule that from here on the word "graph" is to mean a finite graph unless the contrary is stated explicitly.

The terminology of graph theory is not yet standardized. Some authors prefer to use the terms "point" and "line" rather than "vertex" and "edge." This usage may be found inconvenient in problems involving both graphs and geometrical or topological structures. In some of the older papers we may find "branch" used for "edge," and "node" for "vertex."

An edge is called a *link* or a *loop* according as the number of its ends is two or one. We shall however get into the habit of saying that each edge has two ends, with the explanation that in the case of a loop the two ends are coincident. The two ends of an edge are said to be *joined* by that edge, and to be *adjacent*. Accordingly we say that a vertex is joined to itself, or is adjacent to itself, if and only if it is incident with a loop. Two or more links with the same pair of ends are said to constitute a *multiple join*. A graph without loops or multiple joins is called a *strict* graph.

There are many problems of graph theory in which only strict graphs are of interest. Accordingly some authors restrict the term "graph" to mean what we have called a strict graph. When they have occasion to add loops or multiple joins to their structures they speak of "multigraphs."

Examples of graphs are not difficult to find. For one, the edges and vertices of a convex polyhedron are the edges and vertices, respectively, of a graph G. The ends in G of an edge are its ends in the geometric sense. We call G the graph of the polyhedron.

A roadmap can be interpreted as a graph. The vertices are the junctions, and an edge is the stretch of road from one junction to the next, or from a junction back to itself. Similarly an electrical circuit may give us a graph in which the vertices are terminals and the edges wires.

It is not difficult to see graphs in genealogical tables and computer programs. All through mathematics they are visible to the graph-theoretical eye of faith, for much of mathematics can be described in terms of binary relations, and what is a binary relation but a graph?

It is customary to represent a graph G by a drawing on paper. The vertices are drawn as dots. A link with ends x and y is represented by a straight or curved line joining the dots of x and y, and not meeting any other vertex-dot. A loop with end x is drawn as a curve leaving the dot of x and returning to it again, without meeting any other vertex-dot on the way.

In such a drawing it may happen that two edge-curves intersect at some point away from all the vertex-dots. Normally such edge-crossings are ignored as not representing anything in the structure of G. But when we ask what is the least possible number of edge-crossings in a drawing of G, deep and difficult problems arise. (See [8], 122–3.)

Some examples follow. Fig. I.1.1 is a drawing of the graph of a cube, and Fig. I.1.2 shows a graph with loops and multiple joins.

Some graphs with simple structures are thought to deserve special names. For some purposes it is convenient to recognize a *null* graph, having no edges and no vertices. A *vertex-graph* is an edgeless graph having exactly one vertex (Fig. I.1.3(i)). A *loop-graph* consists of a single loop with its one end (Fig. I.1.3(ii)), and a *link-graph* consists of a single link with its two ends (Fig. I.1.3(iii)).

Let n be a nonnegative integer. Then an *n-clique* is defined as a loopless graph with exactly n vertices and $\frac{1}{2}n(n-1)$ edges, each pair of vertices being joined by a single link. Thus the n-cliques are strict graphs. Evidently the null graphs are the 0-cliques, the vertex-graphs are the 1-cliques, and the link-graphs are the 2-cliques. Figure I.1.4 shows a