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This volume contains the proceedings of a conference on Combinatorial Theory that took place at Schloss Rauischholzhausen in May 1982 to mark the 375th anniversary of the Universität Giessen. There were eight invited lectures and over twenty contributed talks. 21 of these papers are contained in this volume. In accordance with the aim of the conference, they cover the whole range of Combinatorics. We hope that the conference and this book will contribute to a better understanding of the various aspects of this fast developing and diverging field, as well as stimulate the exchange of ideas.

We would like to thank all the referees for their cooperation and, in particular, their prompt response. We are also indebted to Frau D. Begemann and to Frau R. Schmidt for helping with the organizational details of the conference, and to the Hochschulgesellschaft for financial support. Finally, we are very grateful to the secretaries of the Mathematisches Institut; without their help, the manuscript would not have been completed in time.

Giessen, October 1982

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CONTENTS

* An asterisk indicates an invited speaker

CRITICAL PERFECT SYSTEMS OF DIFFERENCE SETS WITH THE MINIMUM START

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The concept of a perfect system of difference sets has been introduced in [4] as a mathematical model of the following problem in radio-astronomy: A few movable antennas are used in several successive configurations to measure various spatial frequencies relative to some area of the sky. The distances between antennas determine the frequencies obtained. We do not want to miss any frequency, and want to avoid redundancy (repetition of the same spacing between antennas). For more details, the reader is referred to [4] and [5].

Let $c,m,p_1,\dots,p_m$ be positive integers, let $S_i = \{x_{0i}<x_{1i}<\dots<x_{p_i,i}\}$, $i = 1,\dots,m$ be sequences of integers, and let $D_i = \{x_{ji}-x_{ki},\ 1\le k<j\le p_i\}$, $i = 1,\dots,m$ be their difference sets. Then we say that the system $\{D_1,\dots,D_m\}$ is a perfect system of difference sets (PSDS) starting with c if $\bigcup_{i=1}^{m} D_i = \{c,c+1,\dots,c-1+\sum_{i=1}^{m}\binom{p_i+1}{2}\}$. Each set D_i is called a component of the PSDS $\{D_1,\dots,D_m\}$. The size of D_a is p_a, the half-size of D_a is $r_a = [p_a/2]$ where $[x]$ denotes the integer part of a real number x. Then $p_a = 2r_2+\delta_a$ where $\delta_a = 0$ or 1 according to whether p_a is even or odd. This notation will be used throughout the paper. The reader will observe that the size of a component is not the number of its elements; if the size of D_a is p_a then D_a has $1+2+\dots+p_a = \frac{1}{2}p_a(p_a+1)$ elements.

We will briefly review some earlier results concerning PSDS:

A PSDS is called regular if all its components have the same size. A regular PSDS with m components of size p, starting at c, will be called an (m,p,c)-system. In [4], the existence of (m,p,1)-systems has been related to graceful numberings of certain graphs, and some relations between m,p,c, necessary for the existence of an (m,p,c)-system, have been obtained. Further existence studies have been carried out in [7]; one of the results obtained here is that, if an (m,p,c)-system exists, then $p \le 4$. Without this result, a lot of time and money could have been spent in efforts aimed at finding (m,p,c)-systems with large values of p. A generalization of this result to the nonregular case has been obtained in [9]: Every PSDS contains at least one "small" component (a component of size ≤ 4). This has been further generalized in [2]: Every PSDS starting at c ($c \geq 1$) contains at least c small components. This follows immediately from the inequality (5) below. Proceeding from the inequality (2) it has been proved in [1] that, in a PSDS with m components with the half-sizes $r_1 \leq r_2 \leq \ldots \leq r_m$, it is $r_m \leq K(\sqrt{m}+1)$ where K is a constant, and that the average of half-sizes of the components of any PSDS is bounded by a constant. The first result implies that the number of perfect systems of difference sets starting with a given c, which has a given number m of components, is finite. Moreover, it follows from the results in [2] that $c \leq m$. This means that the number of all PSDS with a given number of components and all possible starts c is finite.

Let us now denote (similarly as in [1, 2])

$$n = \frac{1}{2}\sum_{a=1}^{m}(2r_a+\delta_a)(2r_a+\delta_a+1)$$

$$s = \frac{1}{2}\sum_{a=1}^{m} r_a(3r_a+2\delta_a+1)$$

$$\ell' = \frac{1}{2}\sum_{a=1}^{m} r_a(r_a+1), \qquad \ell = n-\ell'$$

and let $S = \{c, c+1, \ldots, c+s-1\}$, $L = \{c+\ell, \ldots, c+n-1\}$, $M = \{c+s, \ldots, c+\ell-1\}$. Furthermore, let us put $x_{j+k-1,a} - x_{j-1,a} = d_{ja}^{k}$, $j = 1,\ldots,p_a$, $k = 1,\ldots,p_a+1-j$, $a = 1,\ldots,m$. Then the elements of D_a can be represented in the form of a difference triangle

$$\begin{array}{ccccc} & & d_{1a}^{p_a} & & \\ & \cdots\cdots & \cdots\cdots & \cdots\cdots & \\ & d_{1a}^{2} & d_{2a}^{2} & d_{p_a-1,a}^{2} & \\ d_{1a}^{1} & d_{2a}^{1} & \cdots\cdots & & d_{p_a,a}^{1} \end{array}$$

The top (bottom) r_a rows of this triangle will be referred to as its upper (lower) half. Then s and ℓ' denote the number of elements in the lower (upper) halves of all triangles corresponding to $\{D_1,\ldots,D_m\}$, and n denotes the number of all elements in all such triangles. According to Proposition 1.1 in [4] we have

$$\sum_{j=1}^{k} d_{ja}^{p_a+1-k} = \sum_{j=1}^{p_a+1-k} d_{ja}^{k}, \quad k = 1,2,\ldots,r_a, \quad a = 1,\ldots,m$$

Adding over k and a we get

$$(1) \qquad \sum_{a=1}^{m}\sum_{k=1}^{r_a}\sum_{j=1}^{k} d_{ja}^{p_a+1-k} = \sum_{a=1}^{m}\sum_{k=1}^{r_a}\sum_{j=1}^{p_a+1-k} d_{ja}^{k}$$

in words: The sum of all elements in the upper halves of the difference triangles corresponding to $D_1,\ldots,D_m$ is equal to the sum of all elements in the lower halves. If we replace the elements in the lower halves by $c,\ldots,c+s-1$ in the middle rows by $c+s,\ldots,c+\ell-1$, and in the upper halves by $c+\ell,\ldots,c+n-1$, we get from (1) the inequality $\sum_{i=c+\ell}^{c+n-1} i \ge \sum_{i=c}^{c+s-1}$, i.e.

$$(2)\qquad (n+\ell+2c-1)(n-\ell) \ge s(s+2c-1), \quad \text{or} \quad (2n-\ell'+2c-1)\ell' \ge s(s+2c-1)$$

and this is equivalent to the fundamental inequality obtained in [1] (see also [2] or [9]). This inequality has been instrumental in establishing a number of important properties of PSDS; for details and other results on PSDS see e.g. [1], [2], [3], [5], [6], [7], [8], [10], [12], [13], [14].

In [2], the inequality (2) has been used to develop another inequality, easier to use, which will be useful in the proof of Theorem 3 below. Since [2] is not yet available in print we will repeat the main steps here.

Let us consider a PSDS with m components and let c_k denote the number of components of size k, $k \ge 2$. Then $m = \sum_{k\ge 2} c_k$ and we can write

$$(3)\qquad \begin{cases} n = \frac{1}{2}\sum_{k\ge 1} c_{2k}(4k^2+2k) + \frac{1}{2}\sum_{k\ge 1} c_{2k+1}(4k^2+6k+2) \\ s = \frac{1}{2}\sum_{k\ge 1} c_{2k}(3k^2+k) + \frac{1}{2}\sum_{k\ge 1} c_{2k+1}(3k^2+3k) \\ \ell' = \frac{1}{2}\sum_{k\ge 1} c_{2k}(k^2+k) + \frac{1}{2}\sum_{k\ge 1} c_{2k+1}(k^2+k) \end{cases}$$

If we denote, for any positive integer p,

$$\varepsilon_p = \sum_{k\ge 1} k^p c_{2k}, \qquad \omega_p = \sum_{k\ge 1} k^p c_{2k+1}$$

then the second inequality in (2) yields

$$(\varepsilon_1+\varepsilon_2+\omega_1+\omega_2)(4c-2+3\varepsilon_1+7\varepsilon_2+4\omega_0+11\omega_1+7\omega_2) -$$
$$- (\varepsilon_1+3\varepsilon_2+3\omega_1+3\omega_2)(4c-2+\varepsilon_1+3\varepsilon_2+3\omega_1+3\omega_2) \ge 0$$

which implies

$$\varepsilon_1^2-\varepsilon_2^2+\omega_1^2-\omega_2^2+2\varepsilon_1\varepsilon_2+4\varepsilon_1\omega_1+2\varepsilon_1\omega_2-2\varepsilon_2\omega_2+2\omega_0(\varepsilon_1+\varepsilon_2+\omega_1+\omega_2) \ge$$
$$(4c-2)(\varepsilon_2+\omega_1+\omega_2) .$$

Adding $2\varepsilon_1^2$ to both sides we can transform the last inequality into

$$(\varepsilon_1+\varepsilon_2+\omega_1+\omega_2)(\varepsilon_1-\varepsilon_2+\omega_1-\omega_2) + 2\varepsilon_1(\varepsilon_1+\varepsilon_2+\omega_1+\omega_2) + 2\omega_0(\varepsilon_1+\varepsilon_2+\omega_1+\omega_2) \geq 2\varepsilon_1^2-(4c-2)\varepsilon_1+(4c-2)(\varepsilon_1+\varepsilon_2+\omega_1+\omega_2).$$

Dividing this last inequality by $\varepsilon_1+\varepsilon_2+\omega_1+\omega_2$ and denoting

$$\Delta_c = \frac{\varepsilon_1(\varepsilon_1-2c+1)}{\varepsilon_1+\varepsilon_2+\omega_1+\omega_2}$$

we get

$$3\varepsilon_1-\varepsilon_2+2\omega_0+\omega_1-\omega_2 \geq 4c-2+2\Delta_c .$$

Substituting from (3) we can transform the last inequality into

(4) $$c_2+c_3+c_4 \geq 2c-1 + \Delta_c + \sum_{k=4}^{\infty} \frac{1}{2}k(k-3)(c_{2k-1}+c_{2k}).$$

Furthermore,

$$2c-1+\Delta_c = \frac{(2c-1)(\varepsilon_2+\omega_1+\omega_2)+\varepsilon_1^2}{\varepsilon_1+\varepsilon_2+\omega_1+\omega_2} = \frac{(2c-2)(\varepsilon_2+\omega_1+\omega_2)}{\varepsilon_1+\varepsilon_2+\omega_1+\omega_2} + \frac{\varepsilon_1^2+\varepsilon_2+\omega_1+\omega_2}{\varepsilon_1+\varepsilon_2+\omega_1+\omega_2}$$

It is $\varepsilon_1 \leq \varepsilon_1^2$, $\varepsilon_1 \leq \varepsilon_2$, and therefore $\dfrac{\varepsilon_2+\omega_1+\omega_2}{\varepsilon_1+\varepsilon_2+\omega_1+\omega_2} \geq \frac{1}{2}$.

This implies $2c-1+\Delta_c \geq \frac{1}{2}(2c-2)+1 = c$, and (4) yields then

(5) $$c_2+c_3+c_4 \geq c + \frac{1}{2} \sum_{k\geq 4} k(k-3)(c_{2k-1}+c_{2k}) .$$

Since (5) is weaker than (2) it holds as a strict inequality whenever (2) holds as a strict inequality.

Throughout this paper, we will use the symbols C_1, C_2, C_3, C_4 to denote the one-component perfect systems of difference sets represented by the following triangles:

											6					6		
		3				3				4		5			5		4	
C_1:	1		2	C_2:	2		1	C_3:	1		3		2	C_4: 2		3		1 .

A perfect system of difference sets will be called critical if its elements satisfy (2) as an equality. Equivalently, $\{D_1,\ldots,D_m\}$ is a critical PSDS if the elements of L,S,M correspond to the elements in the upper halves, lower halves, and middle rows of the difference triangles corresponding to the components of $\{D_1,\ldots,D_m\}$.

The reader will observe that a PSDS starting with c satisfying (5) as an equality satisfies also (2) as an equality and therefore is critical. A noncritical PSDS starting with c satisfies (5) as a strict inequality.

Only critical PSDS with c = 1 (the minimum start) will be considered in this paper. For $c > 1$ we would get different results. The reader will observe that the above PSDS C_1, C_2, C_3, C_4 are all critical (with $c = 1$).

Let us start our study of critical PSDS by investigating the possible position of 1. We get the following

Lemma 1. Let $\Delta = \{D_1, \ldots, D_m\}$ be a critical PSDS and let D_a be its component containing 1. Then D_a also contains s and s+1. Let $d^1_{ia} = 1$. If the size p_a of D_a is odd then $i \neq r_a+1$ where $r_a = [p_a/2]$. If p_a is even then the sizes of all components of $\{D_1, \ldots, D_m\}$ are even numbers.

Proof. To simplify the notation we will drop the subscript a; hence e.g. d^r_i will stand for $d^{r_a}_{i,a}$ etc. Let $d^1_i = 1$. If $i \leq r+\delta$ (where $\delta = p-2r$) we have $d^{r+1}_i = d^1_i + d^r_{i+1} \geq s+1$ and $d^r_{i+1} \leq s$ and this implies $d^r_{i+1} = s$, $d^{r+1}_i = s+1$. If $i \geq r+1$ we get in a similar way $d^{r+1}_{i-r} = s$, $d^{r+1}_{i-r} = s+1$. If p_a is odd and $i = r+1$ we get $d^r_{r+2} = d^r_1 = s$ and this is impossible. Furthermore, we see from these considerations that D_a contains s,s+1. If p_a is even then $s \in S$, $s+1 \in L$, hence $M = \emptyset$ – there is no component of odd size.

Let $\Delta = \{D_1, \ldots, D_m\}$ be a PSDS and let $n = d^1_{hg}$, where D_g is one of the components. Then we will say that n is represented as the difference v-u in D_g if 1. $v-u = n$ 2. either $d^{r_g}_{h+1,g} = u$, $d^{r_g+1}_{hg} = v$ or $d^{r_g}_{h-r_g,g} = u$ and $d^{r_g+1}_{h-r_g,g} = v$.

Clearly, we must have $u \leq s$, $v \geq s+1$.

Lemma 2. Under the assumptions of Lemma 1, D_a contains 2 and at least one of the numbers s-1, s+2 (in addition to the numbers 1,s,s+1). The numbers 1,2 are neighbors in the first row of the triangle corresponding to D_a if and only if Δ is either C_1 or C_2. If 2 is represented in D_a as the difference of s+2 and s then Δ is either C_3 or C_4.

Proof. Let us denote by D_b the component of Δ containing 2; the subscript b will again be omitted. Let us define j ($1 \leq j \leq p_b$) by $d^1_j = 2$. Then $d^{r+1}_j = 2+d^r_{j+1}$ if $j \leq r+\delta$, $d^{r+1}_{j-r} = d^r_{j-r}+2$ if $j \geq r+1$. The two cases are similar (they can be transformed into each other by writing the elements of each row in the triangle representing D_b in the opposite order), so only the first case will be discussed. Since $d^{r+1}_j \geq s+1$, $d^r_{j+1} \leq s$, we have either $d^{r+1}_j = s+2$, $d^r_{j+1} = s$ or $d^{r+1}_j = s+1$, $d^r_{j+1} = s-1$. This implies that $2 \in D_a$ (since, $s, s+1 \in D_a$) and, therefore, $D_b = D_a$. In the first case, $s+2 \in D_a$, in the second one, $s-1 \in D_a$.

Let us now distinguish the two cases (according to the way in which 2 is represented as the difference of two elements of D_a).

I. Let 2 be represented as the difference of s+2 and s. Let $d_j^1 = 2$, $d_j^{r+1} = s+2$, $d_{j+1}^r = s$ for some $j \le r+\delta$. Let $d_i^1 = 1$. If $i \le r+\delta$ we have $d_{i+1}^r = s$, hence $i = j$, which is impossible, hence $i \ge r+1$, and $d_{i-r}^r = s$; this implies $i = j+r+1$ and $p_a > 2$. Furthermore, $d_{i-r}^{r+1} = d_{j+1}^{r+1} = d_{j+1}^r + d_{j+r+1}^1 = s+1$ and $d_j^{r+2} = d_j^1 + d_{j+1}^{r+1} = s+3$, hence s+3 L and $M = \{s+1, s+2\}$, $p_a = 3$. Let us now denote $L' = L-\{s+3\}$, $S' = S-\{1,2,s\}$. If $L' \neq \emptyset \neq S'$ there exist a $y \in L'$ and an $x \in S'$ such that $y = x+3$ (since 3 must be in some component of our PSDS), and this is impossible, as $y \ge s+4$, $x \le s-1$. We conclude that $L' = \emptyset = S'$ and we have a PSDS with only one component of size 3, which must coincide with either C_3 or C_4.

II. Let 2 be represented as the difference of s+1 and s-1. Let $d_j^1 = 2$, $d_j^{r+1} = s+1$, $d_{j+1}^r = s-1$ for some $j \le r+\delta$. If $d_i^1 = 1$ we can see as in case I that $i \ge r+1$ and $d_{i-r}^r = s$, $d_{i-r}^{r+1} = s+1$ which implies $i = j+r$. If 1,2 were neighbors we would have $r = 1$, $p = 2$ or 3. If $p = 2$, $3 \in L$ and we get C_1 or C_2. If $p = 3$, $3 \in M$ and the first row of the triangle corresponding to D_a must contain a number > 3 which belongs to S- and this is impossible.

<u>Lemma 3</u>. Let $\Delta = \{D_1, \ldots, D_m\}$ be a critical perfect system of difference sets different from each of C_1, C_2, C_3, C_4. Then $3 \in D_a$ (i.e. 3 is in the same component as 1 and 2). Let k be defined by $d_{ka}^1 = 3$. Then either 3 is represented in D_a as the difference of s+3 and s and $k \le r_a + {}_a$ or 3 is represented in D_a as the difference of s+2 and s-1 and $k \ge r_a+1$.

Proof. Let us denote by D_α the component containing 3; the subscript α will again be omitted. Let k be defined by $d_k^1 = 3$ (in D_α); now we have the following three possibilities.

A. $d_{k+1}^r = s$, $d_k^{r+1} = s+3$ (if $k \le r+\delta$)
or $d_{k-r}^r = s$, $d_{k-r}^{r+1} = s+3$ (if $k \ge r+1$)

B. $d_{k+1}^r = s-1$, $d_k^{r+1} = s+2$ (if $k \le r+\delta$)
or $d_{k-r}^r = s-1$, $d_{k-r}^{r+1} = s+2$ (if $k \ge r+1$)

C. $d_{k+1}^r = s-2$, $d_k^{r+1} = s+1$ (if $k \le r+\delta$)
or $d_{k-r}^r = s-2$, $d_{k-r}^{r+1} = s+1$ (if $k \ge r+1$)

Under our assumptions, if D_a contains 1,2, it contains s, s+1, s-1; this means that, in all 3 cases considered here, $3 \in D_a$, hence $D_\alpha = D_a$.

We will now show that the above case C is impossible. In case C, if $k \le r+\delta$, we have $d^r_{k+1} = s-2$, $d^{r+1}_k = s+1$. If $d^1_j = 2$, we have $d^{r+1}_j = s+1$ and we see that $j = k$ which is impossible. If $k \ge r+1$ we have $d^{r+1}_{k-r} = s+1$, hence $k = r+j$. According to the proof of Lemma 1 (Case I), we have also $i = r+j$ where $d^1_i = 1$, hence $i = k$ and this is again impossible.

Let us now consider case A. If $k \ge r+1$, we have $d^r_{k-r} = s$. If i is defined by $d^1_i = 1$ we have by the proof of Lemma 2 (Case I) $i = r+j$, $d^r_{i-r} = s$, hence $i = k$ and this is a contradiction.

In case B, if $k \le r+\delta$, we have $d^r_{k+1} = s-1$; we also have $d^r_{j+1} = s-1$ where $d^1_j = 2$, and this is a contradiction again. This completes the proof of Lemma 2.

<u>Lemma 4</u>. Let Δ satisfy the assumptions of Lemma 3. Let j,k be defined again by $d^1_j = 2$, $d^1_k = 3$ $(j \le k+\delta)$. If 3 is represented as the difference of $s+3$ and s then $k = j-1$. If 3 is represented as the difference of $s+2$ and $s-1$ then $k = j+r+1$.

Proof. We have $d^1_{j+r} = 1$ (see Lemma 2) and $d^r_j = s$, and this implies $j = k+1$ in the first case. In the second case, we have $d^r_{j+1} = s-1 = d^r_{k-r}$ and this implies $k = j+r+1$.

<u>Theorem 1</u>. Let Δ satisfy the assumptions of Lemma 3. Then Δ cannot have any component of odd size.

Proof. Let us assume the opposite. Then, according to Lemma 1, D_a is of odd size. Let us consider two separate cases.

A. 3 is represented as the difference of $s+3$ and s.

Then $d^1_{j-1} = 3$, $d^1_j = 2$, $d^1_{j+r} = 1$, $d^r_j = s$, $d^{r+1}_j = s+1$, $d^{r+1}_{j-1} = s+3$, and $d^{r+2}_{j-1} = d^1_{j-1}+d^{r+1}_j = s+4$. This implies $s+4 \in L$, $M = \{s+1,s+2,s+3\}$, $p_a = 5$. There are two possibilities for the position of $s+2$ in the triangle representing D_a: either $s+2 = d^{r+1}_{j+1}$ or $s+2 = d^{r+1}_{j-2}$. In the first case, $s+2 = d^{r+1}_{j+1} = d^r_{j+1}+d^1_{j+r+1} = s-1+d^1_{j+r+1}$, hence $d^1_{j+r+1} = d^1_{j-1} = 3$ which is impossible. In the second case, $j = 3$, $d^1_2 = 3$, $d^1_3 = 2$, $d^1_5 = 1$, $d^r_j = d^2_3 = s$, $d^3_1 = s+2 = d^1_1+3+2$, i.e. $d^3_1 = s-3$, and $d^2_1 = s-3+3 = s = d^2_3$ which is a contradiction.

B. 3 is represented as the difference of $s+2$ and $s-1$.

Then $d^1_j = 2$, $d^1_{j+r} = 1$, $d^1_{j+r+1} = 3$ (see Lemma 4). We have $d^{r+2}_j = d^1_j+d^{r+1}_{j+1} = 2+s+2 = s+4 \in L$, so again $M = \{s+1,s+2,s+3\}$ with $s+1 = d^{r+1}_j$,

$s+2 = d^{r+1}_{j+1}$. We have two possibilities for the position of $s+3$ this time: either $s+3 = d^{r+1}_{j+2}$ or $s+3 = d^{r+1}_{j-1}$. We observe that $p_a = 5$, $r_a = 2$ again. In the first case, $d^1_j = 2$, $d^1_{j+2} = 1$, $d^1_{j+3} = 3$, and $s+3 = d^3_{j+2} = d^1_{j+2}+d^1_{j+3}+d^1_{j+4} = 4+d^1_{j+4}$ which implies $d^1_{j+4} = s-1 = d^2_{j+1}$ and we have a contradiction. In the second case, $s+3 = d^3_{j-1} = d^1_{j-1}+d^1_j+d^1_{j+1}$. Since $s = d^2_j = d^1_j+d^1_{j+1}$ we have $d^1_{j+1} = s-2$, and we conclude that $d^1_{j-1} = 3 = d^1_{j+3}$ and we have a contradiction again.

Remark. It has been shown in [2] that the average number of differences in the components of any PSDS cannot exceed 21. However, the largest average number of differences ever achieved – to the best of our knowledge – is ten (see [11] and [12]). It was hoped that this can be improved by constructing a (critical) PSDS with one component of size 3 and several components of size 5 [6]. Using a computer, P.J. Laufer attempted to construct such PSDS with up to six components of size 5. All results were negative; Theorem 1 shows why.

To investigate in more detail the last remaining case (when all components are of even size), let us substitute into (2) for n,s,ℓ and write the result as an equality (we consider critical PSDS). We get

$$\Big(\sum_{a=1}^{m} r_a^2\Big)^2 - 2\Big(\sum_{a=1}^{m} r_a^2\Big)\Big(\sum_{a=1}^{m} r_a\Big) - \Big(\sum_{a=1}^{m} r_a\Big)^2 + 2\sum_{a=1}^{m} r_a^2 = 0 .$$

Let us put $x = \sum_{a=1}^{m} r_a$, $y = \sum_{a=1}^{m} r_a^2$. Then the last equation becomes $y^2-2xy-x^2+2y = 0$. Solving for y in terms of x we obtain (since $y \geq 0$)

$$y = x-1 + \sqrt{(x-1)^2+x^2} .$$

Since x,y are positive integers, x must be such that $(x-1)^2+x^2$ is a perfect square. Our search (for $x \leq 120$) provided the following pairs x,y:

$x = 1, \quad y = 1$

$x = 4, \quad y = 8$

$x = 21, \quad y = 49$

$x = 120, \quad y = 169$.

If $x = 1$, $y = 1$, we get either C_1 or C_2. If $x = 4$, $y = 8$, then, necessarily, $m = 2$, and $r_1 = r_2 = 2$. However, no PSDS with 2 components of size 4 can exist (see [7]), Theorem 4.2). For $x = 21$, $y = 49$, our complete search revealed the following possible candidates for critical PSDS (by c_k we denote the number of components of size k, by m the total number of components):

1. $c_2 = 2$, $c_4 = 5$, $c_6 = 3$, $m = 10$
2. $c_2 = 5$, $c_4 = 2$, $c_6 = 4$, $m = 11$
3. $c_2 = 9$, $c_4 = 2$, $c_6 = 0$, $c_8 = 2$, $m = 13$
4. $c_2 = 7$, $c_4 = 2$, $c_6 = 2$, $c_8 = 1$, $m = 12$
5. $c_2 = 11$, $c_4 = 1$, $c_6 = 1$, $c_8 = 0$, $c_{10} = 1$, $m = 14$
6. $c_2 = 8$, $c_4 = 4$, $c_6 = 0$, $c_8 = 0$, $c_{10} = 1$, $m = 13$.

The question is still open whether any of the above 6 possibilities really yields a critical PSDS. At this moment, the problem seems to be too difficult to decide even when using a computer.

The authors are indebted to the referee for pointing out that the above search for x can be replaced by using the Pell equation. The equation $x^2+(x-1)^2 = u^2$ is equivalent to $1 = u^2-2x^2+2x$ or $2 = 2u^2-4x^2+4x$ or $(2x-1)^2-2u^2 = 1$ which is a Pell equation. The Pell equation $M^2-2u^2 = -1$ has the general solution $M = A_k$, $u = B_k$ where $A_k+B_k\sqrt{2} = (1+\sqrt{2})^{2k+1}$. Using this formula we can obtain the same values of x as above.

To summarize, we have:

<u>Theorem 2</u>. There exists no critical PSDS, different from C_1, C_2, C_3, C_4, with the sum of half-sizes ≤ 20. There exists no critical PSDS with the sum of half-sizes equal to 21 and fewer than 10 components. There exist no critical PSDS with the sum of half-sizes between 22 and 119.

We would like to conclude this study by conjecturing that the only critical PSDS with start one are C_1, C_2, C_3, C_4.

The above results yield immediately the following extension of earlier results about the number of small components.

<u>Theorem 3</u>. Every PSDS with start one, which is different from C_1, C_2, C_3, C_4, and such that the sum of the half-sizes of its components is less than or equal to 119, has at least two small components (i.e. components of size 2,3, or 4).

Proof. If our PSDS has c_k components of size k, $k \ge 2$, then it satisfies (5) with $c = 1$:

(6) $$c_2+c_3+c_4 \ge 1 + \frac{1}{2}\sum_{k\ge 4} k(k-3)(c_{2k-1}+c_{2k}) .$$

If the PSDS in question is noncritical, (6) holds as a strict inequality,