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ABSTRACT

In this work we show that almost all useful ordinal invariants in topology studied until now (such as derived length of scattered spaces, sequential order of sequential spaces, etc.) can be brought under the single heading of what we call the order of a map. This helps us to perceive some close connections among apparently unrelated corners of general topology, to view the known concepts from different angles and to obtain a lot of information about the particular cases.

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INTRODUCTION

WHAT IS AN ORDINAL INVARIANT?

Very often, the study of mathematical structures, especially the classification of structural types, is successfully carried out by associating simpler mathematical structures to them. By way of explanation, we mention two instances:

(1) The well-known Ulm's theorem for countable reduced p-groups and its recent generalizations completely solve the classification problem in a fairly large collection of abelian groups, by associating to each group in this collection, a well-ordered transfinite sequence of cardinal numbers.

(2) The dimension of a real vector space is a cardinal invariant which specifies the vector space upto isomorphism.

Apart from such invariants, there are some other invariants, which give elegant classifications of objects in some categories, though they do not classify them completely. We give two examples.

(1) The notions of dimension, weight, density character, suslin number, etc., of topological spaces, provide examples of cardinal invariants in topology.

(2) The homotopy groups, homology groups, etc., are some group-invariants occuring in algebraic topology.

The general set-up is as follows:

Let $\underline{A}$ and $\underline{B}$ be two categories. Let θ be a map which associates to each object of $\underline{A}$ a unique object of $\underline{B}$. Let θ further satisfy the condition: If A_1 and A_2 are isomorphic in $\underline{A}$, then $\theta(A_1)$ and $\theta(A_2)$ are isomorphic in $\underline{B}$. Then θ is called a $\underline{B}$-invariant in $\underline{A}$.

In some nice cases, this θ can be taken as a functor between the two categories. But in several interesting situations, the association of invariants is not of functorial nature.

We stress that the category $\underline{B}$ should be more well-known and more easily manageable than $\underline{A}$; otherwise it is not of much interest. The most suitable candidates for $\underline{B}$ are the simplest ones - e.g., the cardinal numbers, the ordinal numbers and the groups. Of these three, the cardinal numbers and the groups have received much attention in the past. In the recent past, there has been a realization that the ordinal numbers may be equally useful, if not

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more. For instance, we may mention the extension of the notion of Krull dimension as an ordinal number.

Throughout this work, our concern will be with the category TOP of topological spaces, or its subcategories in place of $\underline{A}$. All continuous maps will be taken as morphisms. We shall fix $\underline{B}$ to be the category of ordinal numbers. In other words, we are interested in the ordinal invariants in TOP and its subcategories.

SOME KNOWN ORDINAL INVARIANTS IN TOPOLOGY.

Though not so extensively studied as cardinal invariants, there are some ordinal invariants in topology, that yield significant results. We shall mention some of them in this section, for two reasons. First, they enable one to appreciate the usefulness of ordinal invariants in topology. Secondly, they have close connections with our results, which will be made explicit in the course of this work.

The nicest of the ordinal invariants so far studied in Topology, is the derived length of a scattered topological space. Among the important theorems obtained by the use of this invariant, the most striking one is the following due to MAZURKIEWICZ and SIERPINSKI: Every countable compact Hausdorff space is well-ordered. Further, for each such topological space, there is an associated element of the product set $\omega_1 \times \omega$. This invariant specifies completely the space, among countable compact Hausdorff spaces. This result has been later improved by J. deGroot, by obtaining a similar result for the class of countable locally compact Hausdorff spaces.

The derived length was also used by M. KATETOV to prove the following theorem: A countable regular space admits a continuous bijection onto a compact Hausdorff space if and only if it is scattered. This theorem gives a partial answer to a question of Banach.

Again, it was later used in a paper of the author and M. Rajagopalan to give a direct proof of the fact that every countable scattered metric space can be embedded in a well-ordered space.

Also, BESSAGA and PELCZYNSKI used an ordinal invariant, closely related to the derived length, to give a complete linear topological classification of Banach spaces $C(X)$ for compact scattered metric spaces X.

Thus this invariant has been an efficient tool in several significant results in Topology.

Two equally interesting ordinal invariants in Topology are the sequential order and k-order. These have been studied first by ARHANGELSKII and FRANKLIN and later by many others. Generalizing the former of these, MEYER has studied m-sequential orders. All these form the starting point of our theory. These will be precisely defined and discussed in the fifth chapter.

Defining a closure space as a generalization of a topological space, a natural ordinal invariant in the category of closure spaces has been studied by many. This invariant has close relations with our work and will be considered in the third chapter.

ATTEMPTS OF UNIFICATION.

The similarities among the theories of sequential order, k-order, m-net order, etc., have been so apparent that many have been looking for a unified approach. These theories have been developed separately by different persons in different years; but they have so much in common that their separate treatment ceases to be attractive. To mention an example, the characterization of Fréchet spaces as hereditarily sequential spaces, that of m-Fréchet spaces as hereditarily m-sequential spaces and that of w-Fréchet spaces are hereditarily w-sequential spaces have all been obtained separately in different articles.

This need for unification has already been recognized. For example, this has been emphasized in the introduction of [F-4]. But some caution is necessary here. A blind generalization of these results, is not only not done, but also false: For example, k'-spaces are not the same as hereditarily k-space. Hence, is the difficulty in the formulation of a general theory which is simultaneously unifying and natural.

In the attempts up to now made to unify the approaches under consideration, the following are worth mentioning: FRANKLIN's natural covers and Σ-spaces, MROWKA's R-spaces and MEYER's convergence bases and sub-bases. Of these, the first has been more successful than the others.

In all of these attempts, one finds two essential drawbacks:

(1) None of them would encompass all the mentioned theories as particular cases. Thus for example, a possible untreated approach along the lines of Meyer does not absorb the k-order and the alike; the Σ-characteristic of the theory of natural covers does not absorb the m-net order and the alike. These two approaches give two general theories, each having a sizable hirarchy of particular cases. However, a more general approach that will include all of these, is yet to be found.

(2) Secondly, in each particular case, we see that the ordinal invariant is defined in a coreflective subcategory of TOP. Hence, it is natural to look for one in an arbitrary coreflective subcategory of TOP that would encompass these as particular cases. But such a nicety is not available in these approaches.

These observations make our starting point clear. We consider a general coreflective subcategory of TOP, develop a theory of ordinal invariant that generalizes all the mentioned ones and show that several theorems hold good in the general case also.

We accomplish this and besides we show that the earlier two approaches fit nicely as subcases of our theory of E-order; the former coincides with the case in which E is imagive, the latter corresponds to the case in which every member of E has a unique accumulation point. We further show that our approach has several nice equivalent formulations, that the whole theory has a categorical background and that it produces questions of interest to every topologist.

ADVANTAGES OF OUR APPROACH.

We have mentioned two salient merits of our method, namely completeness and naturality. Apart from these, there are a few other advantages of our method that are worth mentioning here.

Our tool, namely the order of a map, is so fundamental that it connects several corners of general topology that are apparently not much related. For example one of our theorems brings a close connection among the following:

(i) complementation in lattices of topologies;
(ii) extremal epimorphisms in certain pull-back diagrams;
(iii) smallness under a certain ordinal invariant;
(iv) idempotency of compatible Čech closures;
(v) ways of expressing a topology as a meet of other topologies;
(vi) intrinsic conditions of pure topology.

Secondly, the general method immensely enriches the particular cases. For example, the derived length etc., also come under the reign of the order of a map; our method suggests how to extend derived length to non-scattered spaces, how to dualize it etc..

Among the byproducts, a few deserve a mention here. Section 6.2 completely settles a question on hereditarily quotient maps that has been initiated by WHYBURN and considered by some others. Section 4.3 proves a general theorem, a corollary of which answers a question posed by ARHANGELSKII and FRANKLIN. One of the exercises disproves a guess of theirs concerning test spaces for sequential order.

AN OUTLINE OF THIS MONOGRAPH.

This work is self-contained; essentially, everything that is needed for understanding our results, is built within this monograph. A knowledge of elementary concepts of general topology and a bit of ordinal arthmetic, is the only prerequisite assumed on the part of the reader. A separate chapter on preliminaries is dispensed with; the needed concepts are explained then and there.

The work is divided into six chapters of well-defined distinction. The entire study rests on the notion of the order of a map, developed in the first chapter. This ordinal-classification of maps was originally motivated by the desire to obtain characterizations of the following elegant type: Spaces with order $\leq \alpha$ (under some ordinal invariant) are precisely the images of spaces in a certain nice class under certain nice maps.

The rest of the matter can be roughly divided into two parts: The first part consisting of Chapters 2 to 5, deals with E-orders that are defined in a proper subcategory of TOP; the second part consists of a single (but the longest) chapter (Chapter 6), dealing with invariants defined on the whole of TOP.

These two parts are independent to some extent. A large portion of Chapter 6 can be studied after Chapter 1 without going through other chapters, but for some minor definitions.

In the first part, Chapters 2, 3 and 4 develop the general theory of E-order; particular cases, examples and counterexamples are deferred to Chapter 5. In the general theory itself, a three-fold division is made:

(a) introduction and the study of the salient properties of E-order;

(b) establishment of the nicety of this concept, by viewing it in different ways, relating it to known things and furnishing it with categorical flavour;

(c) some problems concerning E-order that are not essential for later study, but are too natural to be excluded.

These respectively constitute Chapters 2, 3 and 4. The last three sections of Chapter 4 can be omitted in the first reading, without affecting the study of later chapters.

The exercises at the end include several closely related results. In spite of their elegance and importance, some results have been deferred to the exercises, in order not to retard the swiftness of our progress towards the main results.

In the course of the work, we come across many results that are more appreciable when seen amidst a group of similar results, than separately. In order to focus them in such a way, they have been classified in some groups and tabulated at the end. These tables and diagrams also help to understand the naturality and a certain amount of completeness in our work.

The final appendix indicates several directions in which these investigations may be worthily continued. It includes many open problems, that may stimulate research in this area.

The author is much indebted to the forerunners in this field, especially to S.P. FRANKLIN, whose works have inspired and initiated him to the present work.

CHAPTER 1

ORDER OF A MAP

In this chapter, we classify maps by associating in a natural way, an ordinal number to each map. The following three questions are considered:

(1) How nicely is this classification related to the basic concepts of general topology?

(2) How does it behave with respect to the familiar operations on maps?

(3) How is it computable in some well-known examples?

1.1. DEFINITION AND ELEMENTARY RESULTS

Let X be any topological space, Y be any set and let f be a function from X onto Y. Using topological structure of X, we now associate an ordinal number $\sigma(f)$ to the function f.

DEFINITION 1.1.1. Let X,Y and f be as above. Let A be any subset of Y. Then take the pre-image of A, take its closure in X and now take the image. We obtain a bigger subset of Y. Repeat this process again and again. More precisely, we let

$$A_f^0 = A \quad \text{and}$$

$$A_f^1 = f(\mathrm{cl}(f^{-1}(A))).$$

If α is an ordinal number and if A_f^β has been defined for every $\beta < \alpha$, then we let

$$A_f^\alpha = \begin{cases} (A_f^\beta)_f^1 & \text{if } \alpha = \beta + 1 \\ \bigcup_{\beta<\alpha} A_f^\beta & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

This defines A_f^α for every ordinal number α. We let

$$\tilde{A}_f = \cup\{A_f^\alpha \mid \alpha \quad \text{is an ordinal number}\}.$$

We define $\sigma(f)$ to be the least ordinal number α such that $\tilde{A}_f = A_f^\alpha$ for every subset A of Y. This $\sigma(f)$ is called the order of the map f.

The following remarks prove the existence of $\sigma(f)$.

REMARK 1.1.2.

(1) As α increases, A_f^α also increases. In other words if $\alpha \leq \beta$, then $A_f^\alpha \subset A_f^\beta$ for each $A \subset Y$.

(2) If for some α, we have $A_f^\alpha = A_f^{\alpha+1}$, then $A_f^\alpha = A_f^\beta$ for every $\beta \geq \alpha$.

(3) $(\tilde{A}_f)_f^1 = \tilde{A}_f$ for every $A \subset Y$.

(4) For a subset B of Y, the following are equivalent:

(a) $B = \tilde{A}_f$ for some $A \subset Y$.

(b) $B = \tilde{B}_f$.

(c) $B = B_f^1$.

(d) $f^{-1}(B)$ is closed.

(5) If α is any ordinal number whose cardinality $|\alpha|$ exceeds $|Y \setminus A|$, then $A_f^\alpha = A_f^{\alpha+1}$.

(6) If $|\alpha| > |Y|$, then $A_f^\alpha = \tilde{A}_f$ for every $A \subset Y$.

(7) The order $\sigma(f)$ always exists and satisfies $\sigma(f) \leq$ the initial ordinal of the cardinal just succeeding $|Y|$.

(8) If $A \subset B \subset Y$, then $A_f^\alpha \subset B_f^\alpha$ for every ordinal number α.

REMARK 1.1.3. Suppose now that Y is also provided with a topology.

(1) If f is continuous, then $A_f^1 \subset \overline{A}$ for every $A \subset Y$.

(2) Conversely if $A_f^1 \subset \overline{A}$ for every $A \subset Y$, then f is continuous.

(3) (By transfinite induction) If f is continuous, then $A \subset A_f^\alpha \subset \overline{A}$ for every $A \subset Y$ and for every ordinal number α.

(4) More generally, the following are equivalent for f:

(a) f is continuous.

(b) $A_f^\alpha \subset \overline{A}$ for every $A \subset Y$ and for some $\alpha \geq 1$.

(c) $\tilde{A}_f \subset \overline{A}$ for every $A \subset Y$.

(d) $A_f^1 = \overline{A}$ for every closed subset A of Y.

(e) $\tilde{A}_f = \overline{A}$ for every closed subset A of Y.

(5) If f is a closed continuous map, then $\overline{A} = A_f^1$ for every $A \subset Y$.

(6) If f is a closed continuous map, then $\sigma(f) \leq 1$.

(7) If we define $\tilde{A}_f$ to be the closure of A for each $A \subset Y$, then

we get another topology on Y.

(8) This "another" topology is exactly the quotient topology on Y with respect to f.

(9) $f: X \longrightarrow Y$ is a quotient map if and only if $\tilde{A}_f = \overline{A}$ for every $A \subset Y$.

(10) If f and g are two maps from a topological space X onto a set Y and if $A_f^1 = A_g^1$ for each subset A of Y, then $\sigma(f) = \sigma(g)$.

(11) For each $A \subset Y$, let

$$\sigma(A,f) = \text{glb}\{\alpha \mid A_f^\alpha = A_f^{\alpha+1}\}.$$

Then

$$\sigma(f) = \text{lub}\{\sigma(A,f) \mid A \subset Y\}.$$

(12) If $f: X \longrightarrow Y$ is a quotient map, then

$$\sigma(f) = \text{glb}\{\alpha \mid A_f^\alpha = \overline{A} \text{ for every } A \subset Y\}.$$

(13) If P is any subset of X, then the f-saturation of P is defined as the bigger set $f^{-1}(f(P))$. We say that P is f-saturated if it is its own f-saturation. If f is continuous, it is true that f takes saturated closed sets in X to closed sets in Y, if and only if f is quotient.

(14) One can formulate an alternate definition of $\sigma(f)$ in terms of the closure of f-saturation.

(15) Let $f: X \longrightarrow Y$ be a quotient map. Then $\sigma(f) = 0$ if and only if Y is discrete.

(16) If, in the definition of $\sigma(f)$, we use the interior operation in the place of closure operation, the process stops always at the second stage and hence we do not get an equivalent definition; however, one arrives at an equivalent definition clearly by relacing in addition, "the saturation" of a set by "the largest saturated subset".

1.2. QUOTIENT MAPS WITH ORDER ≤ 1.

In this section, we give some elegant characterizations of quotient maps with order ≤ 1. The results of this section are very basic in our later discussions.

FIRST TYPE OF CHARACTERIZATION. Let X be a topological space, Y a set and f a function from X onto Y. Then f induces a "critical" topology on Y namely, the quotient topology. To specify it by its open (closed sets, we have: a subset of Y is open (closed) if and only if its pre-image is open (closed) in X. Suppose we want to specify it by "neighborhoods". Then the natural expectation is this: "Let y belong to Y and let V be a subset of Y containing y; then V is a neighborhood of y in Y if and only if $f^{-1}(V)$ is a neighborhood of the set $f^{-1}(y)$ ". But unfortunately, if we define the neighborhood system in Y in this way, not all the axioms are satisfied (i.e., we do not get a topology on Y), unless some conditions are put on f. When we get a topology, it of course coincides with the quotient topology. Therefore, one asks: If $f: X \longrightarrow Y$ is a quotient map, what are the conditions on f to ensure that the topology of Y can be specified directly by neighborhoods, as described above?

Similar is the situation, when we want to describe the closure operation in Y directly in terms of f. A natural description is this: If A is a subset of Y, then the closure of A in Y is just the image of the closure of its preimage . But in general, this operation fails to be idempotent. So, we do not get a topology unless f is of some special type. So one asks: What are the conditions on f so that in the topology of Y, the closure operation can be described directly by f, as mentioned above? Here, we have a simple but interesting result.

<u>PROPOSITION 1.2.1</u>. Let $f: X \longrightarrow Y$ be a quotient map. Then the following are equivalent.

(1) In the topology of Y, the neighborhoods can be described directly by f.

(2) In the topology of Y, the closure can be specified directly by f.

(3) $\sigma(f) \leq 1$.

<u>PROOF</u>. (1) implies (3): Let (1) hold and let $A \subset Y$. We want to show that $A_f^1 = \overline{A}$. Since f is continuous $A_f^1 \subset \overline{A}$. For the reverse inclusion let $y \in \overline{A}$. Suppose there is a neighborhood W of $f^{-1}(y)$ disjoint from $f^{-1}(A)$. Then consider $f(W)$. Its preimage is also a neighborhood of $f^{-1}(y)$. Hence, by (1), $f(W)$ is a neighborhood of y. But since W is disjoint from $f^{-1}(A)$, we get that $f(W)$ is disjoint with A. This contradicts our assumption that $y \in \overline{A}$, thereby proving that every neighborhood of $f^{-1}(y)$ must meet $f^{-1}(A)$. It follows that the closure of $f^{-1}(A)$ must meet $f^{-1}(y)$ and therefore $y \in A_f^1$. Thus we have shown that $A_f^1 = \overline{A}$ for each $A \subset Y$. It follows that $\sigma(f) \leq 1$.

The equivalence of (2) and (3) is obvious, when (2) is reformulated in this way: For each $A \subset Y$, the closure of A coincides with A_f^1.

Now let (3) be true and let us prove (1). If V is a neighborhood of y

in Y, then the continuity of f implies that $f^{-1}(V)$ is a neighborhood of the set $f^{-1}(y)$ in X. Conversely, let V be a subset of Y such that $f^{-1}(V)$ is a neighborhood of $f^{-1}(y)$. Then let $F = X \setminus f^{-1}(V) = f^{-1}(X \setminus V)$. By assumption, the closure of F is disjoint with $f^{-1}(y)$. It follows that y is not in $(X \setminus V)_f^1$ and hence by (3), y is not in $\overline{X \setminus V}$. This means that V is a neighborhood of y.

<u>SECOND TYPE OF CHARACTERIZATION</u>. Now we proceed to show that the class of all quotient maps with order ≤ 1 coincides with some well-known classes. For this, we first observe that a quotient map between two topological spaces, need not be a quotient map when restricted to a subset (even saturated) of its domain. Plenty of examples will be available after the present discussion.

<u>LEMMA 1.2.2</u>. Let X and Y be two topological spaces and let $f: X \longrightarrow Y$ be a continuous onto map.

(a) If $A \subset Y$ is such that whenever $A \subset B \subset \overline{A}$ the restriction of f to $f^{-1}(B)$ is a quotient map then $A_f^1 = A_f^2$.

(b) If $A \subset Y$ is such that $B_f^1 = B_f^2$ for each $B \subset A$, then the restriction of f to $f^{-1}(A)$ is a quotient map.

<u>PROOF</u>. (a) Let the hypothesis of (a) hold and let $x \in \overline{A}$. Take $B = A \cup \{x\}$. By our assumption $f|_{f^{-1}(B)}$ is a quotient map. We want to show that $x \in A_f^1$. If $x \in A$, we are done. If not, $f^{-1}(A)$ cannot be closed in $f^{-1}(B) = f^{-1}(A) \cup f^{-1}(x)$. Hence, $\overline{f^{-1}(A)} \cap f^{-1}(x)$ is nonempty, so that x is in A_f^1. Thus $A_f^1 \supset \overline{A}$ and the assertion follows.

(b) Let the hypothesis of (b) hold and let $B \subset A$ be such that $f^{-1}(B)$ is closed in $f^{-1}(A)$. We want to show that B is closed in A. If $x \in \overline{B} \cap A$, then $x \in \overline{B} = B_f^1$ (by assumption) and hence

$$x \in f(\overline{f^{-1}(B)}) \cap A = f(f^{-1}(B)) \cap A \quad (\text{since } f^{-1}(B) \text{ is closed in } f^{-1}(A))$$
$$= B.$$

This proves that $f|_{f^{-1}(A)}$ is a quotient map.

As an immediate consequence of this lemma, we have

<u>PROPOSITION 1.2.3</u>. The following are equivalent for a quotient map $f: X \longrightarrow Y$:

(1) $\sigma(f) \leq 1$.

(2) For each $A \subset Y$, the restriction of f to $f^{-1}(A)$ is a quotient map.

PROOF. To show (1) implies (2), apply Lemma 1.2.2(b).
To show (2) implies (1), apply Lemma 1.2.2(a).

REMARKS AND DEFINITION 1.2.4.

(a) The maps satisfying condition (2) of the above proposition, are called hereditarily quotient maps.

(b) A. A. Arhangelskii [A-1] defines a map $f: X \longrightarrow Y$ between two topological spaces to be pseudo-open if the following holds: whenever $y \in Y$ and V is a neighborhood of the set $f^{-1}(y)$ in X, it is true that $f(V)$ is a neighborhood of y in Y. Obviously this is a generalization of open maps. [A-1] proves that a continuous map is pseudo-open if and only if it is hereditarily quotient.

(c) A map f between two topological spaces is said to be quasi-closed if the image of the closure of any f-saturated set is closed. Clearly, every closed map is quasi-closed.

(d) Din Ne Tong [D] has defined a class of maps, which he calls "pre-closed maps". He establishes several interesting facts about them concerning monotoneness, extension, etc.. They coincide with the pseudo-open maps defined above.

In terms of these definitions, we have

PROPOSITION 1.2.4. The following are equivalent for a map f from a topological space X onto a topological space Y:

(1) f is pseudo-open and continuous.
(2) f is quasi-closed and continuous.
(3) f is quotient and $\sigma(f) \leq 1$.

PROOF. The equivalence of (1) and (3) is clear, when we notice that (1) is equivalent to the condition (1) of Proposition 1.2.1. The equivalence of (2) and (3) can also be proved easily.

COROLLARY 1.2.5. (a) Every open continuous map has order ≤ 1.
(b) Every closed continuous map has order ≤ 1.

THIRD TYPE OF CHARACTERIZATION. In this paragraph, we obtain a characterization of quotient maps with order ≤ 1, which reveals the fact that this notion is a natural generalization of "being either open or closed".