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Problems in Distributions and Partial Differential Equations

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C. ZUILY

Université Paris Sud
Orsay
France

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INTRODUCTION

The aim of this book is to provide a comprehensive introduction to the theory of distributions through solved problems. It was originally written for undergraduate students in Mathematics but it can be used by a wider audience, engineers, physicists and also by more advanced students.

The first six chapters deal with the classical theory with special emphasis on the concrete aspect. The reader will find many examples of distributions and learn how to work with them.

The last chapter, written for more advanced readers, is a very short introduction to a very wide and important field in analysis which can be considered as the most natural application of distributions, namely the theory of partial differential equations. The reader will find exercises on the classical differential operators (Laplace, heat, wave $\bar{\partial}$, elliptic operators), on fundamental solutions, on hypoellipticity, analytic hypoellipticity, on Sobolev spaces, local solvability, on the Cauchy problem etc. At the beginning of each chapter the theoretical material used in it is briefly recalled. Moreover, the more difficult problems are indicated by one (or more) star(s).

At the end of the book the interested reader will find an index of words, an index of notations and a short bibliography where he will be able to find material for further study.

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CHAPTER 1

Preliminaries

PROGRAMME

Spaces whose topology is defined by a collection of semi-norms

Space $C^k(\Omega)$ $(0 \leq k \leq +\infty)$ of k-times differentiable functions on an open subset Ω of $\mathbb{R}^n$

Space $\mathscr{D}(\Omega)$ (or $C_0^\infty(\Omega)$) of C^∞ functions with compact support in Ω.

The Leibniz formula

The Taylor formula with integral remainder.

BASICS — CHAPTER 1

a) Notations

A multi-index $\alpha \in \mathbb{N}^n$ can be written $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j \in \mathbb{N}$. We shall denote

$$|\alpha| = \alpha_1 + \cdots + \alpha_n; \quad \alpha! = \alpha_1! \cdots \alpha_n!; \quad \alpha - \beta = (\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n);$$

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!}$$

If α and β are two multi-indices in $\mathbb{N}^n$ we write $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ $i = 1, \dots, n$. For $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$ we set $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. Moreover we shall set

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \text{ where } \partial_j = \frac{\partial}{\partial x_j}$$

The expression $P = \sum_{|\alpha| \leq m} a_\alpha(x)\partial^\alpha$ will be called a differential operator of order $m \in \mathbb{N}$ and the functions $a_\alpha(x)$ the coefficients of the operator.
The support of a function f, denoted by supp f, will be the closure of the set $\{x: f(x) \neq 0\}$

b) Spaces whose topology is defined by a collection of semi-norms.

Let E be a vector space on a field K ($\mathbb{R}$ or $\mathbb{C}$). A semi-norm on E is a map p from E to $\mathbb{R}_+ = \{x \in \mathbb{R}, x \geq 0\}$ such that

i) $p(\lambda x) = |\lambda| p(x) \quad \forall x \in E \quad \forall \lambda \in K$

ii) $p(x + y) \leq p(x) + p(y) \quad \forall x \in E \quad \forall y \in E$

We say that p is a norm if moreover $p(x) = 0$ implies $x = 0$.
Let I be a subset of $\mathbb{R}$ and $(p_i)_{i \in I}$ a collection of semi-norms on E. For every $x_0 \in E$, $\varepsilon > 0$ and for all finite subset F of I we set

$$V(x_0, \varepsilon, F) = \{x \in E, p_i(x - x_0) < \varepsilon, i \in F\}$$

The collection $V(x_0, \varepsilon, F)$, when $\varepsilon > 0$ and F ranges over the finite subset of I, defines a filter of neighborhoods of x_0 and thus generates a topology on E which is compatible with the linear structure on E (which means that the maps $(x, y) \mapsto x + y$ from $E \times E$ to E and $(\lambda, x) \mapsto \lambda x$ from $K \times E$ to E are continuous). We say then that E is a locally convex topological vector space (l.c.t.v.s). Let us assume that I is countable (we may take $I = \mathbb{N}$) then the topology defined by the collection $(p_i)_{i\in\mathbb{N}}$ is metrizable. Indeed if for x and y in E we set

$$d(x, y) = \sum_{i=0}^{\infty} \frac{1}{2^i} \cdot \frac{p_i(x - y)}{1 + p_i(x - y)}$$

one can show that d is a distance on E and that the topology defined by d is equivalent to the one defined by the collection $(p_i)_{i\in\mathbb{N}}$.

Let $(E, (p_i)_{i\in I})$, $(F, (q_j)_{j\in J})$ be two l.c.t.v.s. Let T be a linear map from E to F. Then T is continuous if and only if:

For every semi-norm q_j there exists a positive constant C and a semi-norm p_i such that:

$$q_j(Tx) \leqq Cp_i(x)$$

for every $x \in E$.

The reader interested in these questions may consult reference [4].

c) The spaces $C^k(\Omega)$

Let Ω be an open subset of $\mathbb{R}^n$ and $k \in \mathbb{N}$ (or $k = +\infty$). We denote by $C^k(\Omega)$ the space of functions defined in Ω with values in $\mathbb{C}$ which are k times (or infinitely) differentiable. It is equipped with the semi-norms

$$p_K(u) = \sum_{|\alpha|\leq k} \sup_{x\in K} |\partial^\alpha u(x)| \text{ where } K \text{ is a compact subset of } \Omega \text{ (if } k \in \mathbb{N})$$

$$p_{K,j}(u) = \sum_{|\alpha|\leq j} \sup_{x\in K} |\partial^\alpha u(x)| \text{ where } K \text{ is a compact subset of } \Omega \text{ and } j \in \mathbb{N} \text{ (if } k = +\infty)$$

They give the topology of uniform convergence, on every compact, of the derivatives of order less or equal to k (if $k \in \mathbb{N}$) and of all derivatives (if $k = +\infty$).

These topologies are metrizable and then the spaces $C^k(\Omega)$ are complete for $0 \leq k \leq \infty$.

d) The space $\mathscr{D}(\Omega)$ or $C_0^\infty(\Omega)$

It is the space of all C^∞ functions on Ω with compact support. To define the topology of $\mathscr{D}(\Omega)$ one has to introduce the notion of inductive limit topology. The reader may consult [4]. For the sequel it will be sufficient to know how the sequences converge. One has the following result.

A sequence $(\varphi_j)_{j\in\mathbb{N}}$ of elements of $\mathscr{D}(\Omega)$ converges to zero in $\mathscr{D}(\Omega)$ if and only if:

i) There exists a compact subset K of Ω such that for every $j \in \mathbb{N}$, supp $\varphi_j \subset K$.
ii) For every $\alpha \in \mathbb{N}^n$ the sequence $(\partial^\alpha \varphi_j)_{j\in\mathbb{N}}$ converges uniformly in K to zero.

If a is a C^∞ function on Ω the maps $\varphi \mapsto a\varphi$ and $\varphi \mapsto \dfrac{\partial \varphi}{\partial x_j}$ are continuous from $\mathscr{D}(\Omega)$ to itself.

If K is a fixed compact subset of Ω we shall denote by $\mathscr{D}_K(\Omega)$ the space of all u in $\mathscr{D}(\Omega)$ such that supp $u \subset K$.

e) The Leibniz formula

Let u and v be two functions in $C^k(\Omega)$. Then for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq k$ one has

$$\partial^\alpha(u \cdot v) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^\beta u \cdot \partial^{\alpha-\beta} v$$

f) The Taylor formula with integral remainder

Let $x_0 \in \mathbb{R}^n$ and φ be a C^∞ function in a neighborhood of x_0. Then for every $N \in \mathbb{N}$ and all x in a neighborhood of x_0 we have

$$\varphi(x) = \sum_{|\alpha| \leqslant N} \frac{1}{\alpha!}(x - x_0)^\alpha \cdot (\partial^\alpha \varphi)(x_0) +$$

$$+ \int_0^1 (1 - t)^N \sum_{|\alpha| = N+1} \frac{N+1}{\alpha!}(x - x_0)^\alpha (\partial^\alpha \varphi)(tx + (1 - t)x_0)\, \mathrm{d}t$$

g) The polar coordinates in $\mathbb{R}^n$.

They are defined for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $(r, \theta_1, \ldots, \theta_n) \in]0, \infty[\times]0, \pi[\times \cdots \times]0, \pi[\times]0, 2\pi[$, by the formulas

$$\begin{cases} x_1 = r \cos \theta_1 \\ x_2 = r \sin \theta_1 \cos \theta_2 \\ x_{n-1} = r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \cos \theta_{n-1} \\ x_n = r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-1} \end{cases}$$

Then we have $dx = r^{n-1}(\sin \theta_1)^{n-2}(\sin \theta_2)^{n-3} \cdots \sin \theta_{n-2}\, \mathrm{d}r\, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{n-1}$. We shall write shortly $x = r \cdot \omega$, $\omega = (\omega_1, \ldots, \omega_n)$, and one can see that $|\omega| = 1$, which means that ω belongs to the unit sphere S^{n-1}. Then $\mathrm{d}x = r^{n-1}\, \mathrm{d}r\, \mathrm{d}\omega$ where $\mathrm{d}\omega$ is the measure on S^{n-1}. If $f \in L^1(\mathbb{R}^n)$ one can write

$$\int_{\mathbb{R}^n} f(x)\, \mathrm{d}x = \int_0^\infty \int_{S^{n-1}} f(r \cdot \omega) r^{n-1}\, \mathrm{d}r\, \mathrm{d}\omega$$

STATEMENTS OF EXERCISES* CHAPTER 1

Exercise 1: Borel's theorem

Our purpose is to prove that given a sequence $(a_j)_{j\in\mathbb{N}}$ of complex numbers there exists a function $f \in C^\infty(\mathbb{R})$ such that $\left[\frac{d^j}{dx^j} f\right](0) = a_j, j = 0, 1, 2, \ldots$

a) Let $\varphi \in \mathscr{D}(]-2, 2[)$ such that $\varphi = 1$ for $|x| \le 1$.
Prove that we can find a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of real numbers such that if we set

$$(1) \quad f_n(x) = \frac{a_n}{n!} x^n \varphi(\lambda_n x)$$

then

$$(2) \quad \sup_{x\in\mathbb{R}} \left|\left(\frac{d}{dx}\right)^k f_n(x)\right| \le 2^{-n} \quad \text{for } 0 \le k \le n-1$$

b) Prove that the series $\sum_{n=0}^{\infty} f_n(x)$ defines a function $f(x)$ which is C^∞ and solves our original problem.

Exercise 2

Let Ω be an open set of $\mathbb{R}^n$, k and m be two positive integers such that $k \geqq m$ and $P(x, \partial) = \sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha$ be a differential operator of order m whose coefficients are in $C^{k-m}(\Omega)$.
Prove that P is continuous from $C^k(\Omega)$ in $C^{k-m}(\Omega)$.

Exercise 3

Prove that there is no function δ in $C_c^0(\mathbb{R})$ (the space of continuous functions with compact support) (resp. in $L^1(\mathbb{R})$) such that $\delta * f = f$ for all f in $C_c^0(\mathbb{R})$ (resp. in $L^1(\mathbb{R})$).
(Hint: Use the equality $f(0) = \int f(x)\delta(-x)\,dx$).

Exercise 4

Let $\varphi \in \mathscr{D}(\mathbb{R})$ and $M > 0$ such that $\operatorname{supp} \varphi \subset \{x \in \mathbb{R}: |x| \le M\}$. If $n \in \mathbb{N}$ we set

$$\psi(x) = \begin{cases} \dfrac{1}{x^{n+1}}\left\{\varphi(x) - \sum_{j=0}^{n} \dfrac{x^j}{j!}\varphi^{(j)}(0)\right\} & \text{for } x \ne 0 \\ \dfrac{1}{(n+1)!}\varphi^{(n+1)}(0) & \text{for } x = 0 \end{cases}$$

*Solutions pp 18 to 24