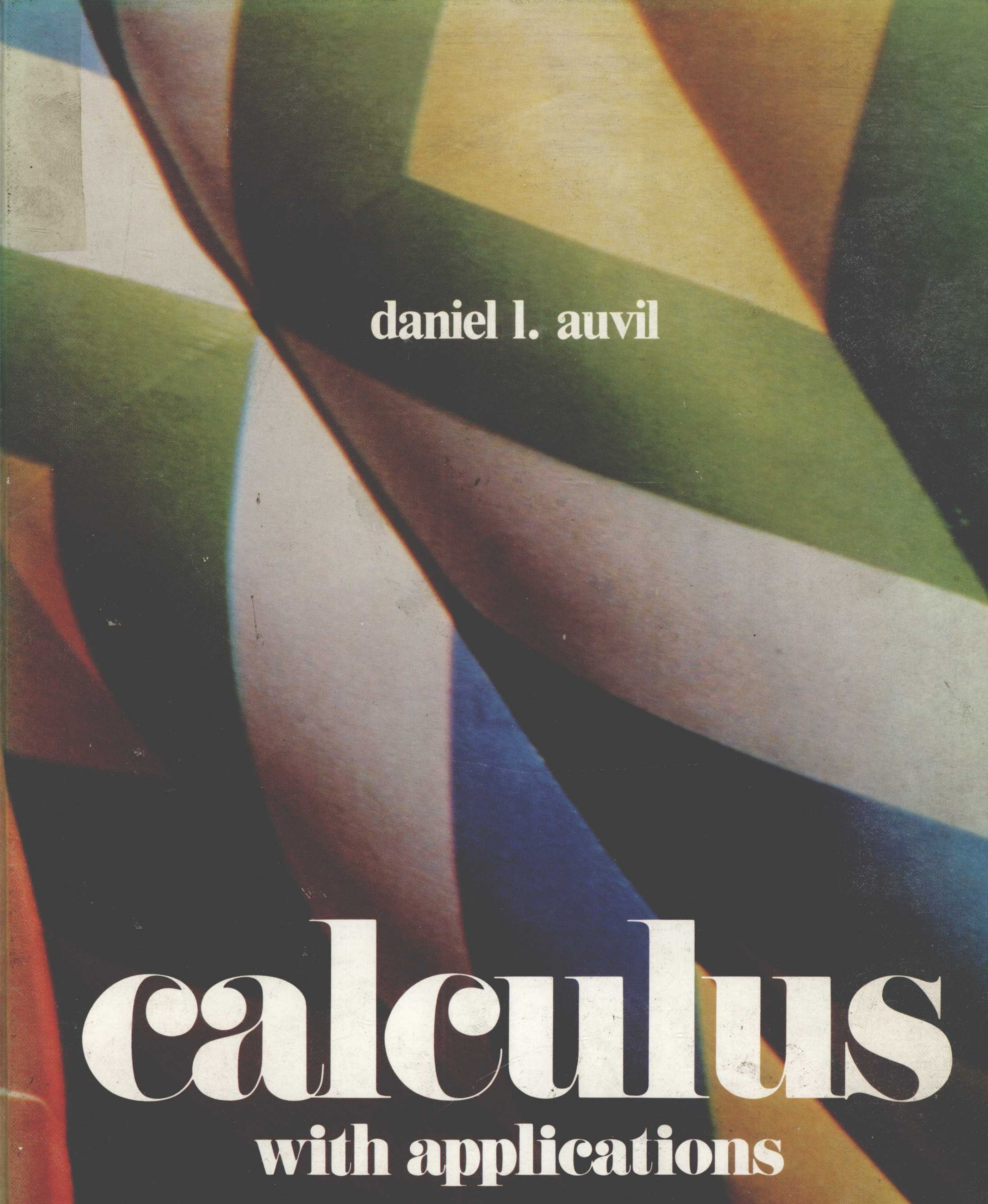

daniel l. auvil

calculus
with applications

daniel l. auvil
Kent State University

addison-wesley publishing company
Reading, Massachusetts • Menlo Park, California
London • Amsterdam • Don Mills, Ontario • Sydney

calculus with applications

Sponsoring Editor: Steven Quigley
Production Editor: Laura Lane
Designer: Marie McAdam
Illustrator: PanTek
Cover Design: Richard Hannus
Cover Photography: Four by Five, Inc.

Library of Congress Cataloging in Publication Data

Auvil, Daniel L.
Calculus, with applications.

Includes index.
1. Calculus. I. Title.
QA303.A84 515 81-14914
ISBN 0-201-10063-0 AACR2

ISBN 0-201-10063-0
ABCDEFGHIJ-HA-898765432

preface

A quote found in a university student calendar went something like this:

If I had only one day left to live,
I would live it in my calculus
class—it would seem
so much longer.

The message, of course, is clear. It emphasizes the problem of constructing a calculus text that is rigorous enough to be worthwhile and meaningful, while at the same time is intuitive enough to be fresh and interesting.

Like the other books I have written, I have tried to make this one as readable as possible. The language is clear and direct. The style is warm and informal. The approach is intuitive and geometric. Whenever appropriate, a picture is used to help clarify the mathematical concepts. I have found that students can enjoy a mathematics course if they are able to read and understand the text and, having done so, are able to work the problems.

It is also important to students that they see how the mathematical ideas they are learning can be used to solve real-world problems. For that reason, almost every section in the text contains a variety of meaningful applications. Since many students who use this text are not mathematics majors, many of these applications are drawn from the management, social, and life sciences.

To further increase understanding and motivation, new topics are normally introduced using specific examples. There are, in fact, more than 350 examples in the text. Only after the mechanics of the examples are understood is a general rule formulated.

To reinforce and review concepts, a spiral approach has been used throughout the text. This allows the student to experience the same topic several times, each time from a slightly different perspective and in slightly greater detail.

The strength of any text, of course, lies in its problems. Each problem set is clearly divided into three parts: regular problems, challenge problems, and calculator problems.

The regular problems consist of those problems normally encountered in a text such as this. Although not labelled as such, these problems are themselves divided into three categories. They begin with straightforward problems that are designed to get students going while they are still in the process of assimilating the new ideas that were just presented in the section. After these initial few problems, subsequent problems are slightly more difficult and are designed to provide drill in the techniques of the section. Finally, the regular problems finish with still more difficult problems that stimulate thinking and interest and provide applications to the ideas of the section. To facilitate use of the text by the instructor, the regular problems are matched so that each odd-numbered problem has an even-numbered counterpart. That is, in each set of exercises, Problem 1 is similar to Problem 2, Problem 3 is similar to Problem 4, and so on. A solid course in calculus can be taught using assignments only from the regular problems.

For exceptional classes, an optional set of challenge problems follows each set of regular problems. A few of the challenge problems are no more difficult than the regular problems; they simply investigate a different aspect of the section being studied. On the other hand, some of the challenge problems are quite difficult. In some cases instructors may wish simply to present the solution to a particular challenge problem themselves, rather than make it a class assignment.

Following each set of challenge problems there is a set of calculator problems.[†] These problems will enable students to become familiar with the use of the hand-held calculator. They involve the same concepts as the regular problems and the challenge problems, but they require more complex computations. A scientific calculator with keys for the following operations is recommended.

y^x	$\sqrt[x]{y}$	e^x	$\ln x$	log	sin	cos	tan

At the end of each chapter there is a set of chapter review problems. These problems are not identified with a particular section, hence they provide students with a means of reviewing and testing their understanding of the material in the entire chapter. The chapter review problems are on the same level as the regular problems. That is, they are not as difficult as the challenge problems and they do not involve the laborious computations of the calculator problems.

In all there are more than 3500 problems. The quantity and variety of the problems allow the instructor the flexibility to design the level of the course he or she wishes to teach.

---

† Except for a few cases where the regular problems themselves dictate the use of a calculator.

Answers to the odd-numbered problems, as well as answers to all of the chapter review problems, are provided at the end of the text. In addition, a student supplement is available that contains worked-out solutions to all of the even-numbered problems.

Most texts of this type devote only a cursory review to those topics that are normally covered in a previous mathematics course; namely, the topics of algebra, exponential and logarithmic functions, and trigonometry. It has been my experience, however, that there is a wide range in the mathematics backgrounds of students who enroll in a course such as this one. Therefore a thorough treatment of these topics is presented in the text. Algebra is reviewed in Sections 1.1 through 1.6, exponential and logarithmic functions in Sections 5.1 through 5.4, and trigonometry in Sections 9.1 through 9.5. Of course, each instructor knows his or her own class best, so it will be left up to individual instructors as to how involved they want to become in reviewing these topics. Some instructors may wish to omit these sections entirely.

As with any book, this book is not the accomplishment of just one individual. Many others must share the credit for its successful completion. The most notable among these are: John Bishir, North Carolina State University at Raleigh; Suzanne Butschun, Tacoma Community College; J. Myron Hood, California Polytechnic State University–San Luis Obispo; Harold Huneke, University of Oklahoma; George W. Peglar, Iowa State University; and Harold D. Shane, Bernard Baruch College of The City University of New York who reviewed the manuscript. Also, a special note of thanks to my good friends and able colleagues Bernard Richards and Laura McGregor, and to Julie Froble for her expert typing of the final manuscript.

One final note to the student who uses this text. A mathematics text is not a novel. It must be read much more slowly, much more carefully, and with a pencil and paper nearby. Keeping this in mind, I know that you will discover as I have that the independent mastery of even a single problem of good mathematics is an immensely satisfying intellectual activity of the very finest kind.

January 1982
North Canton, Ohio D.L.A.

1 algebra review

2 limits, continuity, and tangent lines

3 differentiation

4 applications of the derivative

5 exponential and logarithmic functions

6 integration

7 methods of integration

8 calculus of several variables

9 the trigonometric functions

appendixes

1 2 3
4 5 6
7 8 9

algebra review

1.1 EXPONENTS AND RADICALS

The study of calculus requires a solid background in the fundamentals of algebra. The purpose of this chapter is to provide a review of those fundamentals, with particular attention given to the specific tools needed throughout the remainder of this text.

One topic that receives a great deal of attention in calculus is the notion of an exponent. Exponential notation enables us to write certain types of algebraic expressions in a more convenient form. *Positive-integer exponents* are defined as follows.

Definition Let a be any real number and n any positive integer. Then

$$a^n = a \cdot a \cdot a \cdots a,$$

where there are n factors of a.

The number a in the definition above is called the **base**, and the number n is called the **exponent**. The expressions a^2 and a^3 are read *a squared* and *a cubed*, respectively. In general, the expression a^n is read *a to the nth power*.

EXAMPLE 1 $(-5)^3 = (-5)\cdot(-5)\cdot(-5) = -125.$

EXAMPLE 2 $x^3 = x \cdot x \cdot x.$

EXAMPLE 3

$$\begin{aligned}(x^2+1)^3 &= (x^2+1)(x^2+1)(x^2+1)\\ &= (x^4+2x^2+1)(x^2+1)\\ &= x^6+x^4+2x^4+2x^2+x^2+1\\ &= x^6+3x^4+3x^2+1.\end{aligned}$$

Note that the definition above does not apply when $n = 0$. We define a *zero exponent* as follows.

Definition If $a \neq 0$, then $a^0 = 1$.

EXAMPLE 4 $2001^0 = 1.$

EXAMPLE 5 $(y^5 - 32)^0 = 1, \quad \text{if } y^5 - 32 \neq 0.$

Negative-integer exponents are defined in terms of positive-integer exponents.

Definition Let a be any real number except zero and n any positive integer. Then $-n$ is a negative integer and

$$a^{-n} = \frac{1}{a^n}.$$

EXAMPLE 6 $10^{-2} = \frac{1}{10^2} = \frac{1}{100}.$

EXAMPLE 7 $\frac{1}{x^{-3}} = \frac{1}{\frac{1}{x^3}} = x^3, \quad \text{if } x \neq 0.$

EXAMPLE 8 $(x + 4)^{-1} = \frac{1}{x + 4}, \quad \text{if } x \neq -4.$

EXAMPLE 9 $10(3t - 1)^{-2} = \frac{10}{(3t - 1)^2}, \quad \text{if } t \neq \frac{1}{3}.$

Rational exponents are defined in terms of radicals. Therefore, before we define rational exponents, we shall recall the definition of $\sqrt[n]{a}$.

Definition If a is a *nonnegative* real number and n is a positive integer, then

$$\sqrt[n]{a} = b,$$

where b is a real number such that $b^n = a$ and $b \geq 0$. If a is a *negative* real number and n is an *odd* positive integer, then

$$\sqrt[n]{a} = b,$$

where b is a real number such that $b^n = a$ and $b < 0$.[†]

The expression $\sqrt[n]{a}$ is called the **principal nth root** of a. The symbol $\sqrt{}$ is a **radical sign**, the number a is called the **radicand**, and the number n is called the **index**. When the index is 2, it is generally not written. Thus $\sqrt{a}$ denotes the principal *square root* (or *second root*) of a. On the other hand, $\sqrt[3]{a}$ denotes the principal *cube root* (or

---

† In this case, $b < 0$ by necessity.

third root) of a. Table 1 in the Appendix contains square roots and cube roots for the integers from 1 to 150.

EXAMPLE 10 $\sqrt{81} = 9$, since $9^2 = 81$, and $9 \geq 0$.

EXAMPLE 11 $\sqrt[3]{-64} = -4$, since $(-4)^3 = -64$.

Note that we have not defined $\sqrt[n]{a}$ for the case where a is a negative real number and n is an *even* positive integer. That is, $\sqrt{-36}$ and $\sqrt[4]{-1}$ have not been defined.

EXAMPLE 12 $\sqrt{5x + 17}$ is defined when $5x + 17 \geq 0$, that is, when $x \geq -\frac{17}{5}$.

EXAMPLE 13 $\sqrt{t^2 + 1}$ is defined for all real numbers t, since $t^2 + 1 \geq 0$ for all real t.

We are now ready to define rational exponents.

Definition Let a be any real number, m any integer, and n any positive integer. Then if m/n is in lowest terms[†]

$$a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m},$$

assuming $\sqrt[n]{a}$ is defined.

EXAMPLE 14 Calculate $16^{3/4}$.

$$16^{3/4} = (\sqrt[4]{16})^3 = 2^3 = 8, \quad \text{or}$$
$$16^{3/4} = \sqrt[4]{16^3} = \sqrt[4]{4096} = 8.$$

In this example it was more convenient to use the form $a^{m/n} = (\sqrt[n]{a})^m$ when calculating $16^{3/4}$. On other occasions it may be more convenient to use the form $a^{m/n} = \sqrt[n]{a^m}$.

The definition for negative-integer exponents applies to rational exponents as well.

EXAMPLE 15 Write $5^{-2/3}$ in radical form.

$$5^{-2/3} = \frac{1}{5^{2/3}} = \frac{1}{\sqrt[3]{5^2}} = \frac{1}{\sqrt[3]{25}}.$$

† If m/n is not in lowest terms, we simply reduce it until it is.

EXAMPLE 16 Write $(x^2 + 4)^{-1/2}$ in radical form.

$$(x^2 + 4)^{-1/2} = \frac{1}{(x^2 + 4)^{1/2}} = \frac{1}{\sqrt{x^2 + 4}}.$$

We have now extended our definition of exponents so that any exponent that is a rational number is defined. In Chapter 5 we shall extend the notion of an exponent even farther to include *irrational exponents*.

Rational exponents have been defined in such a way that they obey five *laws of exponents*, which are stated below.

Laws of Exponents Let a and b be any real numbers and r and s any rational numbers. Then:

1. $a^r \cdot a^s = a^{r+s}$,
2. $\dfrac{a^r}{a^s} = a^{r-s}$,
3. $(a^r)^s = a^{rs}$,
4. $(ab)^r = a^r b^r$,
5. $\left(\dfrac{a}{b}\right)^r = \dfrac{a^r}{b^r}$,

assuming the expressions involved are defined.

EXAMPLE 17 Find the product $\sqrt{x}(\sqrt{x^3} + 6x)$.

Changing to fractional exponents and multiplying, we have

$$\begin{aligned}\sqrt{x}(\sqrt{x^3} + 6x) &= x^{1/2}(x^{3/2} + 6x)\\ &= x^{1/2} \cdot x^{3/2} + x^{1/2} \cdot 6x\\ &= x^{1/2+3/2} + 6x^{1/2+1}\\ &= x^2 + 6x^{3/2}.\end{aligned}$$

If $r = 1/n$ where n is a positive integer, the two laws of exponents $a^r b^r = (ab)^r$ and $a^r/b^r = (a/b)^r$ are often stated in radical form.

Laws of Radicals Let a and b be any real numbers and n any positive integer. Then:

1. $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$,
2. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$,

assuming the expressions involved are defined.

These laws of radicals can sometimes be used to simplify radical expressions.

EXAMPLE 18 Write $\sqrt{24x^3y^4}$ in simplest form. Assume both variables are positive.

$$\sqrt{24x^3y^4} = \sqrt{4 \cdot 6 \cdot x^2 \cdot x \cdot y^4} = \sqrt{4x^2y^4} \cdot \sqrt{6x} = 2xy^2\sqrt{6x}.$$

EXAMPLE 19 Write $(a^{1/3}b^{-2/3})^6(a^{-1/2}b^{-4/3})^{-1}$ in simplest form with positive exponents.

$$\begin{aligned}(a^{1/3}b^{-2/3})^6(a^{-1/2}b^{-4/3})^{-1} &= (a^{1/3})^6(b^{-2/3})^6(a^{-1/2})^{-1}(b^{-4/3})^{-1}\\ &= a^2b^{-4}a^{1/2}b^{4/3}\\ &= a^{5/2}b^{-8/3}\\ &= \frac{a^{5/2}}{b^{8/3}}.\end{aligned}$$

EXAMPLE 20 Write $\dfrac{(x+5)^2}{\sqrt[3]{x}}$ as a sum of terms in powers of x.

$$\begin{aligned}\frac{(x+5)^2}{\sqrt[3]{x}} &= \frac{x^2+10x+25}{x^{1/3}}\\ &= \frac{x^2}{x^{1/3}} + \frac{10x}{x^{1/3}} + \frac{25}{x^{1/3}}\\ &= x^{2-1/3} + 10x^{1-1/3} + 25x^{-1/3}\\ &= x^{5/3} + 10x^{2/3} + 25x^{-1/3}.\end{aligned}$$

EXAMPLE 21 Rationalize the numerator of $\dfrac{\sqrt{x}-3}{x-9}$.

We multiply numerator and denominator by $\sqrt{x}+3$.

$$\begin{aligned}\frac{\sqrt{x}-3}{x-9} &= \frac{\sqrt{x}-3}{x-9} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3}\\ &= \frac{x+3\sqrt{x}-3\sqrt{x}-9}{(x-9)(\sqrt{x}+3)}\\ &= \frac{x-9}{(x-9)(\sqrt{x}+3)}\\ &= \frac{1}{\sqrt{x}+3}.\end{aligned}$$

In Section 5.2 we shall prove that the amount A_t in a savings account after t years is given by the formula

$$A_t = A_0(1 + r)^t,$$

where A_0 is the initial amount and r is the annually compounded interest rate *expressed as a decimal.*

EXAMPLE 22 Suppose \$2500 is invested in an account paying 8% interest compounded annually. How much money is in the account after 5 years?

Using the formula given above with $A_0 = 2500$, $r = 0.08$, and $t = 5$, we have

$$\begin{aligned} A_5 &= 2500(1 + 0.08)^5 \\ &= 2500(1.08)^5 \\ &\approx 2500(1.469328)^\dagger \\ &= \$3673.32. \end{aligned}$$

In the problems that follow we shall assume for simplicity that the variables involved represent only those real numbers for which the expressions in the problems are defined.

PROBLEMS

Expand each expression.

1. 6^3 **2.** 5^4 **3.** $(-3x)^5$ **4.** $(-2y)^7$ **5.** $(x + h)^2$ **6.** $(y - h)^2$ **7.** $(x^2 + 5)^3$ **8.** $(y^3 - 2)^3$

Write each expression in simplest form with positive exponents.

9. $5 \cdot 4^{-3}$ **10.** $\dfrac{3}{8^{-2}}$ **11.** $\left(\dfrac{1}{3}\right)^{-2}$ **12.** $\left(\dfrac{1}{5}\right)^{-3}$

13. $\dfrac{10^0}{(y + 1)^{-2}}$ **14.** $11^0(x - 1)^{-3}$ **15.** $x^{-5}y^5$ **16.** $\dfrac{x^{-4}}{y^4}$

17. $\dfrac{x^n}{x}$ $(n > 1)$ **18.** $x^n \cdot x$ $(n \geq 0)$ **19.** $15(5t - 3)^{-1}$ **20.** $12(3t + 2)^{-1}$

21. $(x + y)(x^{-1} + y^{-1})$ **22.** $(x + 2)^{-7}(x - 3)^{-7}$ **23.** $\dfrac{1}{(x + 4)^{-8}(x + 3)^{-8}}$ **24.** $\left(\dfrac{a^2b^{-1}}{c^{-3}}\right)^{-3}$

---

† This was computed using a calculator with a $\boxed{y^x}$ key.

Find each root. Assume all variables represent positive real numbers.

25. $\sqrt{121}$ **26.** $\sqrt[3]{-27}$ **27.** $-\sqrt[3]{-8}$ **28.** $-\sqrt[4]{625}$ **29.** $\sqrt[5]{-x^5y^{15}}$

30. $-\sqrt[5]{-x^{10}y^{10}}$ **31.** $\sqrt{x^2+2x+1}$ **32.** $\sqrt{x^2+4x+4}$ **33.** $\sqrt{x^{2n}y^{4m}}$ **34.** $\sqrt{x^{2n+2}}$

Calculate each of the following.

35. $64^{2/3}$ **36.** $(-32)^{3/5}$ **37.** $(-27)^{-2/3}$ **38.** $81^{-1/4}$

Perform the indicated operations and express each answer using positive exponents.

39. $x^{7/2}\cdot x^{-1/2}$ **40.** $\dfrac{x^{3/4}}{x^{1/4}}$ **41.** $(x+7)^{1/2}(x+7)^{-3/2}$

42. $(x^2+1)^{1/3}(x^2+1)^{-4/3}$ **43.** $\dfrac{x^{1/3}x^{1/4}y^{-1/2}}{x^{1/12}y}$ **44.** $\dfrac{x^{-1/2}y^{1/2}y^{1/5}}{xy^{1/5}}$

45. $\sqrt[4]{x^3}(x^{1/4}+x)$ **46.** $\dfrac{1}{\sqrt[3]{x}}(x^{1/3}+x)$ **47.** $(a^{7/2}b^{-3/2})^4(a^{-1/2}b^3)^{-1}$

48. $(a^{5/2}b^{1/3})^{-6}(a^2b^{-1/2})^{-1}$ **49.** $\left[\dfrac{(x^2+2)^{3n}}{(y^2+2)^{2n}}\right]^{1/6}$ $(n<0)$ **50.** $\left[\dfrac{(x-3)^{n/2}}{(y-3)^{n/3}}\right]^6$ $(n<0)$

Write each expression using rational exponents.

51. $\sqrt{x^2+4}$ **52.** $\sqrt[3]{x^3-8}$ **53.** $\dfrac{\sqrt{(x+1)^3}}{x-2}$ **54.** $\dfrac{x+4}{\sqrt{(x-3)^5}}$ **55.** $\dfrac{6}{x\sqrt[3]{x}}$

56. $\dfrac{5}{x\sqrt{x}}$ **57.** $\sqrt{1+\sqrt{x}}$ **58.** $\sqrt{x^2+\sqrt{1+x^2}}$ **59.** $\dfrac{\sqrt{x^2-1}}{\sqrt{x+1}}$ **60.** $\sqrt{x-1}\sqrt{x+1}$

Write as a radical in simplest form.

61. $(48)^{1/2}$ **62.** $(125)^{1/2}$ **63.** $\sqrt[3]{27x^4y^5}$ **64.** $\dfrac{\sqrt[3]{48x^8y^5z}}{\sqrt[3]{3x^2y}}$

Write as a sum and/or difference of terms in powers of x.

65. $x^2-\dfrac{1}{\sqrt{x^3}}+\dfrac{8}{x}$ **66.** $x+\dfrac{1}{\sqrt[3]{x}}-\dfrac{10}{x^2}$ **67.** $\dfrac{x(x-2)}{x^2}$

68. $\dfrac{3x(1+x)}{x^2}$ **69.** $\dfrac{(\sqrt{x}+2)^3}{\sqrt{x}}$ **70.** $\dfrac{(\sqrt{x}-1)^3}{\sqrt[3]{x^2}}$

Rationalize the numerator.

71. $\dfrac{\sqrt{x}-2}{x-4}$ **72.** $\dfrac{\sqrt{x}-5}{x-25}$ **73.** $\dfrac{\sqrt{t}-\sqrt{x}}{t-x}$

74. $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$ **75.** $\sqrt{x+1}-\sqrt{x}$ **76.** $\sqrt{x^2+x}-x$