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Günter Schwarz

Hodge Decomposition – A Method for Solving Boundary Value Problems

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Introduction

The central theme of this work is the study of the Hodge decomposition of the space $\Omega^k(M)$ of differential forms on manifolds with boundary, mainly under analytic aspects. In the boundaryless (compact) case, the Hodge theory is a standard tool for characterising the topology of the underlying manifold. It has far reaching implications in complex analysis and algebraic geometry. For manifolds with non-vanishing boundary, results on the connection between the Hodge decomposition on the one hand, and cohomology theory, on the other, are also well established.

Our interest in studying the Hodge decomposition of differential forms is of analytical rather than of topological nature, and restricted to the case of real differentiable manifolds with boundary. We will employ such kind of decompositions, in order to solve boundary value problems for differential forms. This approach is in the spirit of Helmholtz [1858]. He first formulated a result on the splitting of vector fields into vortices and gradients, which can be understood as a rudimentary form of what is now called the "Hodge decomposition". We will show that on the basis of an appropriate decomposition result for the space $\Omega^k(M)$ of differential forms, a variety of linear boundary value problems are solvable in a direct and elegant way.

To formulate a motivating example, we consider differential forms $\omega \in \Omega^k(M)$ of degree k as anti-symmetric k-tensors. The space $\Omega^1(M)$ can be identified with the space $\Gamma(TM)$ of vector fields on M. Restricting ourselves – for this particular example – to the case of a domain $G \subset \mathbb{R}^3$ with smooth boundary ∂G, we consider the boundary value problem

$$\begin{aligned} \operatorname{curl} X &= W \qquad \text{on } G \\ X|_{\partial M} &= 0 \qquad \text{on } \partial G \end{aligned} \tag{0.1}$$

Precise knowledge about the solvability of this problem is of crucial interest e.g. in the Navier-Stokes theory. To formulate necessary and sufficient conditions for the existence of solutions of (0.1), we assume that we can make sense of the direct decomposition

$$\Gamma(TG) = \{ \operatorname{curl} U \mid (\operatorname{curl} U)^{\parallel} = 0 \} \oplus \{ \widetilde{U} \mid \operatorname{curl} \widetilde{U} = 0 \} \tag{0.2}$$

of the space of vector fields into the space of "curls" and the space of curl-free fields. The space of curls is chosen such that its elements obey the boundary condition $(\operatorname{curl} U)^{\parallel} = 0$, i.e. are have a vanishing component parallel to ∂G. The splitting (0.2) is a variant of the celebrated Helmholtz decomposition of vector fields, and follows from the Hodge decomposition for manifolds with boundary as a special case. (In the boundaryless case $G = \mathbb{R}^3$ the corresponding result is known as the "fundamental theorem of potential theory".)

In order to solve the boundary value problem (0.1) we decompose the prescribed vector field W according to the splitting (0.2), yielding

$$W = \operatorname{curl} U_W + \widetilde{U}_W \qquad \text{where} \qquad (\operatorname{curl} U_W)^{\parallel} = 0 \text{ and } \operatorname{curl} \widetilde{U}_W = 0 \ .$$

Obviously, the vanishing of the component $\widetilde{U}_W$ is a necessary condition for solving (0.1). To see that this is also sufficient, we employ the Ansatz $X = U_W + \operatorname{grad} g$ for the solution, where $g \in C^{\infty}(G)$ is to be specified. Since $\operatorname{curl}(\operatorname{grad} g) = 0$, this solves the boundary value problem in view, if and only if one can choose the function g in such a way that

$$(\operatorname{grad} g)^{\parallel} = -(\operatorname{curl} U_W)^{\parallel} \qquad \text{and} \qquad (\operatorname{grad} g) \cdot \mathcal{N} = 0 \text{ on } \partial G \ . \tag{0.3}$$

Here $\mathcal{N}$ is the unite normal field on the boundary ∂G. In fact, by using the collar theorem – as a tool from the theory of manifolds – the extension problem (0.3) is solvable for each vector field $\operatorname{curl} U_W$. This implies the solvability of (0.1) under the integrability condition $\widetilde{U}_W = 0$ for the curl-free component of the prescribed vector field $W \in \Gamma(TG)$.

After this rough illustration of a special application of the Hodge decomposition on a bounded domain in $\mathbb{R}^3$, we turn towards a proper formulation of that decomposition result in a general context. We will develop the theory for smooth, Riemannian manifolds with boundary, rather than restricting ourselves to (bounded) domains $G \subset \mathbb{R}^n$. Let $\Omega^k(M)$ be the space of smooth differential forms of degree k over a ∂-manifold M, then one has the exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ and the co-differential $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ acting as natural differential operators. (These operators correspond to the curl and divergence in vector analysis.) Where the boundary is concerned, we let $\jmath : \partial M \hookrightarrow M$ be the natural inclusion. Then we speak about a differential form ω with "vanishing tangential component", iff $\mathbf{t}\omega := \jmath^*\omega = 0$. Vice versa, ω has "vanishing normal component", iff $\mathbf{n}\omega := \omega - \mathbf{t}\omega = 0$.

Using this terminology, the relevant subspaces of $\Omega^k(M)$ for the Hodge decomposition are the spaces of exact and co-exact k-forms with vanishing tangential and normal components, respectively, and the space of harmonic fields on M. These are given by :

$$\begin{aligned}
\mathcal{E}^k(M) &:= \{\, d\alpha \mid \alpha \in \Omega^{k-1}(M) \text{ with } \mathbf{t}\omega = 0 \,\} \\
\mathcal{C}^k(M) &:= \{\, \delta\beta \mid \beta \in \Omega^{k+1}(M) \text{ with } \mathbf{n}\omega = 0 \,\} \\
\mathcal{H}^k(M) &:= \{\, \lambda \in \Omega^k(M) \mid d\lambda = 0 \text{ and } \delta\lambda = 0 \,\} \ .
\end{aligned}$$

To specify the functional analytic setting, we extend the space $\Omega^k(M)$ of (smooth) differential forms to appropriate Sobolev spaces. $\Omega^k(M)$ is equipped with an inner product

$$\ll \omega, \eta \gg \; = \int_M \omega \wedge \star\eta \ . \tag{0.4}$$

where $\star : \Omega^k(M) \to \Omega^{n-k}(M)$ is the Hodge operator and $\wedge$ the exterior multiplication. We denote the L^2-completion of $\Omega^k(M)$ with respect to that inner product by $L^2\Omega^k(M)$, and identify – in slight abuse of notion – the subspaces $\mathcal{E}^k(M)$, $\mathcal{C}^k(M)$ and $\mathcal{H}^k(M)$ with their respective L^2-completions. The central result of this work is the generalisation of the classical Hodge decomposition theorem holding for compact manifolds without boundary to ∂-manifolds. The idea goes back to Friedrichs [55] and Morrey [56] :

Theorem (Hodge-Morrey-Friedrichs Decomposition)
Let M be a compact Riemannian manifold with boundary.
(a) The space $L^2\Omega^k(M)$ of square integrable k-forms on M splits into the direct sum

$$L^2\Omega^k(M) = \mathcal{E}^k(M) \oplus \mathcal{C}^k(M) \oplus \mathcal{H}^k(M) \tag{0.5}$$

of the spaces of exact, and co-exact forms (with the prescribed boundary behaviour), and the space of harmonic fields.
(b) The spaces $\mathcal{H}^k(M)$ of harmonic fields in $\Omega^k(M)$ can respectively be decomposed into

$$\mathcal{H}^k(M) = \{\lambda \in \mathcal{H}^k(M) \mid \mathbf{t}\,\lambda = 0\} \oplus \{\kappa \in \mathcal{H}^k(M) \mid \kappa = \delta\gamma\} \tag{0.6a}$$
$$\mathcal{H}^k(M) = \{\lambda \in \mathcal{H}^k(M) \mid \mathbf{n}\,\lambda = 0\} \oplus \{\kappa \in \mathcal{H}^k(M) \mid \kappa = d\epsilon\} \;. \tag{0.6b}$$

Furthermore, the Hilbert space decompositions (0.5) and (0.6a-b), respectively, are L^2-orthogonal with respect to the inner product (0.4).

Under the identification between $\Omega^1(M)$ and $\Gamma(TM)$ this result covers, in particular, the Helmholtz decomposition (0.2). (The correspondence between vector fields and differential forms will be considered in detail in Section 3.5.) For various applications it is essential to have decomposition results also for differential forms of Sobolev class $W^{s,p}$. This is of particular importance, if the boundary value problem in view originates from a non-linear dynamical systems, which one intends to solve e.g. by semi-group methods. In view of this, we will establish the following regularity result, which cannot be found in the literature in that generality :

Regularity Theorem
Let $W^{s,p}\Omega^k(M)$ be the space of differential forms of Sobolev class $W^{s,p}$ – where $s \geq 0$ and $1 < p < \infty$ – and let $W^{s,p}\mathcal{E}^k(M)$, $W^{s,p}\mathcal{C}^k(M)$ and $W^{s,p}\mathcal{H}^k(M)$ denote the completions of the corresponding subspaces of $\Omega^k(M)$ in the $W^{s,p}$-norm. Then the decomposition

$$W^{s,p}\Omega^k(M) = W^{s,p}\mathcal{E}^k(M) \oplus W^{s,p}\mathcal{C}^k(M) \oplus W^{s,p}\mathcal{H}^k(M) \tag{0.7}$$

is direct, algebraically and topological. It is L^2-orthogonal, if $p \geq \frac{2n}{n+2s}$.
The decompositions (0.6a-b) generalise accordingly.
If $\omega \in W^{s,p}\Omega^k(M)$, then the Hodge components $d\alpha_\omega \in W^{s,p}\mathcal{E}^k(M)$ and $\delta\beta_\omega \in W^{s,p}\mathcal{C}^k(M)$ can be chosen such, that one can estimate

$$\| \alpha_\omega \|_{W^{s+1,p}} \leq C_a \| \omega \|_{W^{s,p}} \qquad \text{and} \qquad \| \beta_\omega \|_{W^{s+1,p}} \leq C_b \| \omega \|_{W^{s,p}} \;.$$

These results, the decomposition and the corresponding regularity theorem, constitute the proper framework for solving boundary value problems for differential forms, by generalising the projection techniques discussed exemplarily in the context of Eq. (0.1). We will study this in detail in Chapter 3 of this work, after establishing the Hodge-Morrey-Friedrichs decomposition in Chapter 2.

Historical credits

As the pioneer results on the Hodge decomposition, one has to mention, besides Helmholtz [1858], the works of Hodge [33,41], de Rham [31,55] and Weyl [40]. Hodge and de Rham established the connection between cohomology theory and harmonic differential forms with preassigned periods. Weyl, on the other hand, introduced the method of orthogonal projection in that context, and applied this to solve boundary value problems in the Euclidean space. Later, Kodaira [49] extended these decomposition results to differential forms on a general compact Riemannian manifold.

On compact manifolds with boundary, the decomposition of $L^2\Omega^k(M)$ goes back to Friedrichs [55] and Morrey [56,66]. Friedrichs generalised Weyl's method of orthogonal projection onto the respective components of $L^2\Omega^k(M)$. His approach, however, is based on certain density assumptions for differential forms with prescribed boundary behaviour, which are a-priori far from being obvious. In turn, Morrey proves the decomposition theorem by means of a variational method, i.e. by minimising the "Dirichlet integral" on $\Omega^k(M)$. As an essential ingredient he needs an inequality of Gaffney [51].

Regularity results for the Hodge decomposition are well-established on manifolds without boundary. For ∂-manifolds, however, less is known. Friedrichs and Morrey formulate their decompositions also for differential forms of Sobolev class $W^{1,2}$. The general case has been considered by Morrey but the result was never published – according to an oral communication by J. Marsden.

Where boundary value problems for differential forms are concerned, the results of Friedrichs and Morrey were influenced by Duff and Spencer [52], Duff [55] and Conner [56]. Pickard [83] generalises that approach and studied problems for the case of a non-smooth boundary. The idea to investigate boundary value problems systematically on the basis of the decomposition technique goes back to Morrey. This method has been used as a powerful tool in fluid mechanics by Ebin and Marsden [70], and applies later also to other fields of applied mathematics.

On the other hand, Kress [72] studied related boundary value problems for differential forms on $G \subset \mathbb{R}^n$, by using concepts from classical potential theory. These methods have been applied by von Wahl [90b] to solve the particular problem (0.1) for vector fields on $G \subset \mathbb{R}^3$. This problem has also been considered by Borchers and Sohr [90], and earlier – under topological restrictions on G – by Ladyženskaja and Solonnikov [78], and Bogovskiĭ [79]. For a previous contribution of the author see Schwarz [94].

Decomposition results for non-compact manifolds without boundary have been considered by Cantor [81]. These rely on the choice of appropriately weighted Sobolev spaces which were invented in the early 70ies, and are studied in detail by Lockhart and McOwen [85]. Where boundary value problems for differential forms (respectively for vector fields) in the non-compact case are concerned, a relevant result can be found in Mäulen [75].

The study of the de Rham cohomology of manifolds with boundary goes back to Duff and Spencer [52]. Corresponding Hodge decomposition results are well established, cf. Gilkey [84] for a general reference. These rely on the choice of suitable boundary conditions for the forms in $\mathcal{H}^k(M)$, cf. also Cheeger [80]. The results, however, are not equivalent to the Hodge-Morrey-Friedrichs decomposition, given above. The decomposition (0.6) has been considered under topological aspects also by Wenzelburger [94]. An alternative approach was given by Brüning and Lesch [92] who applied the machinery of Hilbert complexes to the de Rham complex, in particular for the case of manifolds with boundary. Decomposition results for differential forms on ∞-dimensional vector spaces are considered in Arai and Mitoma [91].

The proof of the Hodge-Morrey-Friedrichs decomposition theorem which we will present, is based on the use of functional analytic methods in the theory of partial differential equations. Such concepts were developed around 1960 by Nirenberg, Peetre, Schechter and others. The regularity results which we will prove, rely in a crucial way on the theory of elliptic boundary value problem on vector bundles, as presented in Palais [65] and Hörmander [85].

About the contents

In Chapter 1 we introduce the basic notions needed to prove the Hodge-Morrey-Friedrichs decomposition for manifolds with boundary. Since this links the geometry of manifolds with the theory of partial differential equation, and since we intend to make our results comprehensible for readers working in either of these fields, the presentation is given in some detail.

Sections 1.1 and 1.2 are concerned with the analysis on a Riemannian manifolds with boundary and the calculus of differential forms $\omega \in \Omega^k(M)$ on such ∂-manifolds. Particular attention is paid to the description of the differential forms on the boundary ∂M. Corresponding results can rarely be found in the literature.

In Section 1.3 we consider the Sobolev theory of sections in vector bundles over M, and of differential forms in particular. Roughly spoken, all classical results from the Sobolev theory of functions on $\mathbb{R}^n$, generalise to such section spaces, as long as M is compact. Some arguments from functional analysis, which are elementary but not commonly stated in that form in the literature, are given in Section 1.5.

Section 1.6 is devoted to the notion of ellipticity. We formulate the condition of Lopatinskiĭ-Šapiro on the ellipticity of boundary value problems for vector bundles over ∂-manifolds, and quote fundamental theorems for elliptic operators.

In particular we establish the ellipticity of a special problem for the Laplace operator on $\Omega^k(M)$, namely

$$\begin{aligned} \Delta\omega &= \eta && \text{on } M \\ \mathbf{t}\,\omega = 0 \quad \text{and} \quad \mathbf{t}\,\delta\omega &= 0 && \text{on } \partial M \ . \end{aligned} \tag{0.8}$$

Chapter 2 constitutes the central part of this work. There we study the decomposition results of Morrey and Friedrichs for ∂-manifolds in detail. We start in Section 2.1 with a generalisation of Stokes' theorem, and establish Green's formula, reading as

$$\ll d\omega, \eta \gg \, = \, \ll \omega, \delta\eta \gg + \int_{\partial M} \mathbf{t}\,\omega \wedge \star\mathbf{n}\,\eta \ .$$

The basic tool for proving the decomposition theorem is the Dirichlet integral on $\Omega^k(M)$, which is defined as the bilinear form

$$\begin{aligned} &\mathcal{D} : W^{1,2}\Omega^k(M) \times W^{1,2}\Omega^k(M) \longrightarrow \mathbb{R} \\ &\mathcal{D}(\omega,\eta) \;= \, \ll d\omega, d\eta \gg + \ll \delta\omega, \delta\eta \gg \ . \end{aligned}$$

If the tangential component $\mathbf{t}\omega$ of $\omega \in \Omega^k(M)$ vanishes on the boundary ∂M, then the $W^{1,2}$-norm can be estimated by Gaffney's inequality as

$$\| \omega \|^2_{W^{1,2}} \le C_G \Big(\| \omega \|^2_{L^2} + \mathcal{D}(\omega,\omega) \Big) \ . \tag{0.9}$$

In Section 2.2 we study the space $\mathcal{H}^k_D(M)$ of harmonic fields which have a vanishing tangential component, and its L^2-complement in the space of differential forms which also obey $\mathbf{t}\omega = 0$ on ∂M. We denote this complement by $\mathcal{H}^k_D(M)^{\ominus}$, and employ functional analytic arguments to show :

- The space $\mathcal{H}^k_D(M)$ is finite dimensional.
- The bilinear form $\mathcal{D}$ is $W^{1,2}$-elliptic on the complement $\mathcal{H}^k_D(M)^{\ominus}$.
- For each k-form $\eta \in \mathcal{H}^k_D(M)^{\ominus}$, there exists a unique element $\phi_D \in \mathcal{H}^k_D(M)^{\ominus}$ – called the Dirichlet potential of η – such that

$$\ll \eta, \xi \gg \, = \mathcal{D}(\phi_D, \xi) \qquad \forall \xi \in \Omega^k(M) \text{ with } \mathbf{t}\,\xi = 0 \ . \tag{0.10}$$

By applying Green's formula to (0.10), the Dirichlet potential ϕ_D becomes a weak solution of the elliptic boundary value problem (0.8). Correspondingly we can construct a Neumann potential ϕ_N, obeying the boundary condition $\mathbf{n}\phi_N = 0$.

In Section 2.3 we consider the regularity of these solutions. The problem (0.8) is not covered by standard results for systems of boundary value problems, as e.g. presented in Agmon, Douglas and Nirenberg [64]. Therefore we give an explicit proof, to establish ϕ_D also as a strong solution. On the basis of this, the general elliptic theory implies the regularity result :

- ϕ_D is of Sobolev class $W^{s+2,p}$, if and only if η is of Sobolev class $W^{s,p}$.

Making use of these preliminary observations, we establish the Hodge-Morrey-Friedrichs decomposition in Section 2.4. By specifying a Dirichlet potential ϕ_D associated to $\omega \in \Omega^k(M)$, the Hodge component $d\alpha_\omega \in \mathcal{E}^k(M)$ is determined by $\alpha_\omega = \delta\phi_D$. Correspondingly, the Hodge component $\delta\beta_\omega \in \mathcal{C}^k(M)$ is given via $\beta_\omega = d\phi_N$. The proof of the decomposition theorem then splits into 4 steps :

- *Orthogonality* : The spaces $\mathcal{E}^k(M)$, $\mathcal{C}^k(M)$ and $\mathcal{H}^k(M)$ are mutual orthogonal.
- *Algebraic decomposition* : Each $\omega \in L^2\Omega^k(M)$ splits into $\omega = d\alpha_\omega + \delta\beta_\omega + \kappa_\omega$, with Hodge components $d\alpha_\omega \in \mathcal{E}^k(M)$, $\delta\beta_\omega \in \mathcal{C}^k(M)$, and κ_ω in the L^2-orthogonal complement $(\mathcal{E}^k(M) \oplus \mathcal{C}^k(M))^\perp$.
- *L^2-closedness* : The spaces $\mathcal{E}^k(M)$ and $\mathcal{C}^k(M)$ are closed.
- *Harmonic fields* : The complement $\left(\mathcal{E}^k(M) \oplus \mathcal{C}^k(M)\right)^\perp$ coincides with the space $\mathcal{H}^k(M)$ of harmonic fields.

We furthermore establish regularity results for the decomposition, which follows as a direct consequence of the ellipticity of the boundary value problem (0.8) for the Dirichlet potential and the corresponding problem for Neumann potential.

In Section 2.6 we consider – in brevity – the connection between the Hodge-Morrey-Friedrichs decomposition theorem and cohomology theory. The space $\mathcal{H}^k_D(M)$ is shown to be isomorphic to the k$^{\text{th}}$ relative cohomology $\mathbf{H}^k_r(M)$ of M. We then can state a generalised version of Bochner's theorem on "curvature and Betti numbers" for ∂-manifolds.

The reader may have missed – until now – our comments to Sections 1.4 and 2.5. There we study the Hodge-Morrey-Friedrichs theorem in the non-compact case. In view of applications it is of interest to have access to such results in particular for exterior domains. The proof of the decomposition theorem, however, relies on Rellich's compact embedding theorem for the Sobolev spaces $W^{1,2}\Omega^k(M) \hookrightarrow L^2\Omega^k(M)$, which fails, if the volume of M is infinite.

In Section 1.4 the concept of weighted Sobolev spaces is introduced. We restrict ourselves to $G \subset I\!R^n$, in order to avoid technical conditions on the geometry of M. For weighted Sobolev spaces, denoted by $W^{s,p}_a\Omega^k(G)$, a generalisation of Rellich's lemma was shown by Lockhart [81]. A weighted generalisation of Poincaré's inequality, then allows us to reformulate the estimate (0.9).

The precise formulation of the weighted decomposition result is given in Section 2.5. For the proof, one has, roughly spoken, to replace the spaces $L^2\Omega^k(M)$ and $W^{1,2}\Omega^k(M)$ by appropriately weighted Sobolev spaces and repeat the reasoning of Sections 2.2 to 2.4. The Hodge-Morrey-Friedrichs decomposition theorem then can be shown for the completion in the L^2_1-norm. Regularity theorems are given for differential forms of Sobolev class $W^{s,p}_1$.

In an appendix to section 2, which was contributed by J. Wenzelburger, the behaviour of the decomposition is studied with respect to the deformation of the Riemannian structure on M. It is shown that the decomposition results depend smoothly – in the Frechét sense – on the metric g chosen on the manifold M.

Finally, Chapter 3 is devoted to the study of boundary value problems for differential forms, in the spirit indicated in the discussion of our motivating example (0.1). In Section 3.1 we are concerned with the Dirichlet problem for the exterior derivative, given by

$$d\omega = \chi \quad \text{on } M \qquad \text{and} \qquad \mathbf{t}\,\omega = \mathbf{t}\,\psi \quad \text{on } \partial M \ . \tag{0.11}$$

We give necessary and sufficient integrability conditions and prove Sobolev estimates. In Section 3.2 we generalise that result by considering a corresponding problem with $d\omega$, $\delta\omega$ and $\mathbf{t}\omega$ prescribed. As another variant, we study in Section 3.3 the exterior derivative under general inhomogeneous boundary condition, i.e. the problem

$$d\omega = \chi \quad \text{on } M \qquad \text{and} \qquad \omega|_{\partial M} = \psi|_{\partial M} \quad \text{on } \partial M \ . \tag{0.12}$$

This problem – which is the general version of the example (0.1) – is solvable under the same integrability conditions as imposed on the Dirichlet problem (0.11). One should note that this boundary value problem is not elliptic (!), so standard techniques do not apply.

Section 3.4 is devoted to studying the space of harmonic fields from the point of view of boundary value problems. We show that $\mathcal{H}^k(M)$ is infinite dimensional, and differs – if $\partial M \neq \emptyset$ – from the space of harmonic forms $\theta \in \Omega^k(M)$, which are characterised by $\Delta\theta = 0$. Furthermore we consider the Poisson equation $\Delta\omega = \rho$ on $\Omega^k(M)$ under various boundary conditions. Finally, in Section 3.5, we establish the equivalence between problems for differential forms (of degree 1) and corresponding problems for vector fields.

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Chapter 1

Analysis of Differential Forms

1.1 Manifolds with boundary

The de Rham-Hodge theory of differential forms and the theory of elliptic boundary value problems rest upon on the interaction between the global topological structure of the spaces under consideration and local analytic properties of the prescribed data. The natural framework to describe such an interaction is the concept of manifolds with boundary. These look locally like a finite dimensional half space $\mathbb{R}^n_+$ and at the same time include all global topological information about the domain in view.

In order to give a proper definition, let $\mathbf{u} \neq 0$ be a fixed vector in $\mathbb{R}^n$ and define the corresponding real half space by $\mathbb{R}^n_{\mathbf{u}} = \{\mathbf{x} \in \mathbb{R}^n \mid \; <\mathbf{x}, \mathbf{u}> \; \geq 0\}$. A map $h : \mathbb{R}^n_{\mathbf{u}} \to \mathbb{R}^n$ is defined to be differentiable, if it has a differentiable extension $\widetilde{h} : \mathbb{R}^n \to \mathbb{R}^n$. The derivative Dh of h (in boundary points) is correspondingly defined by the restriction of $D\widetilde{h}$ to the half space $\mathbb{R}^n_{\mathbf{u}}$. Obviously this definition is independent of the choice of the extension. Referring, with respect to the analytic properties of the half space and the fundamental topological concepts, to the literature (e.g. Abraham, Marsden and Ratiu [88], Warner [83], Hirsch [76] or Lang [72]), we can establish the concept of manifolds with boundary.

Definition 1.1.1

Let M be a paracompact topological Hausdorff space and $(U_a)_{a\in A}$ an open (locally finite) covering of M. A homeomorphism

$$\varphi_a \; : \; U_a \to \mathbb{R}^n_{\mathbf{u}_a} \qquad a \in A$$

onto an open subset in $\mathbb{R}^n_{\mathbf{u}_a}$ is called a chart. The corresponding atlas on M, $\mathcal{A}_M = (U_a, \varphi_a)_{a\in A}$, is called of class C^k, if for all $a, b \in A$ the transformations

$$\varphi_b \circ \varphi_a^{-1} \; : \; \varphi_a\,(U_a \cap U_b) \to \mathbb{R}^n$$

are C^k-mapping from $\mathbb{R}^n_{\mathbf{u}_a}$ to $\mathbb{R}^n$. Then M (or more explicitly the pair $(M, \mathcal{A}_M)$) defines an n-dimensional C^k-manifold with boundary. The boundary of M is

$$\partial M := \{\, p \in M \mid \exists \;\text{ chart } \varphi_a \text{ with } \; <\varphi_a(p), \mathbf{u}_a> \; = 0 \,\} \quad . \tag{1.1}$$