Proceedings of the XVII International Conference on

Differential Geometric Methods in Theoretical Physics

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Editor

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PREFACE

This series of conferences on the use of mathematical methods in physics has evolved over the seventeen years of its existence according to those physical theories most in vogue, and has consequently emphasized those mathematical methods of most value in their elucidation. Originally concentrating on differential geometric methods, algebraic results have also played a prominent role. Of particular interest in the present XVII Conference is the attention devoted to the so-called Yang-Baxter Algebra and Quantum Groups; no fewer than 7 of the 32 papers presented here deal with some aspect of that relatively new topic. Quantum Field Theory, Gauge Theory and Strings, continue to engage mathematical physicists; other physical phenomena are represented here, including statistical mechanics and soliton theory. The presentation of novel mathematical techniques, supermanifolds and other geometrical methods, auger well for the continued ingenious application of mathematics to the range of physical phenomena as described in this series of meetings.

In any conference as homogeneous as this one, there must of necessity be a large overlap of subject matter. I have more or less arbitrarily grouped the papers into the following sections –

– Yang-Baxter Algebra and Quantum Groups
– Quantum Field Theories
– Gauge Theories and Strings
– Supermanifolds
– Mathematical Techniques
– Specific Physical Problems.

ACKNOWLEDGEMENTS

The ancient Roman town of Chester on the River Dee in the County of Cheshire (in Latin, Castra Devana- camp on the Dee) was the picturesque site of the 17th International Conference on Differential Geometric Methods in Theoretical Physics. Chester is the only city in England to have its walls intact — and its situation on the border of Wales permitted excursions to the 13th Century Castles of Conway and Ruthin where the Conference Banquet was held, as well as an excursion on the River.

The sessions took place in Chester College, a Teacher's Training College associated with the University of Liverpool, whose material assistance the organisers gratefully acknowledge. The organisation was in the capable hands of Dr Fred Bloore, ably assisted by his wife Joan, as well as Dr Nigel Backhouse and Miss Jane Watson who acted as Conference Secretary. The Committee of the Mathematical Physics Group of the Institute of Physics supplied moral backing to the venture, and financial help was contributed by the S.E.R.C and the University of Liverpool. The Organisers gratefully acknowledge the support of these bodies and the host university, without which this delightful Conference could not have taken place. They are also grateful to the organizers of previous conferences in this series for their advice. I should like to thank Miss Rita Quill of the Open University for help with the editing of this Volume.

(Open University) April, 1989 Allan I. Solomon

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I. Yang-Baxter Algebra and Quantum Groups

YANG-BAXTER ALGEBRAS , INTEGRABLE QFT AND CONFORMAL MODELS

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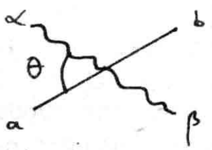
ABSTRACT:
Integrable massive QFT and conformal invariant models follow from lattice integrable models in suitable scaling limits . There are Yang-Baxter algebras (YBA) associated to all these two-dimensional models. These YBA allow to construct the exact solution (spectrum, S-matrix, form-factors,...) for this class of theories.Braid groups and quantum groups are derived as limiting cases of YBA when θ (spectral parameter) goes to $\pm\infty$.

Let us start by defining a Yang-Baxter algebra (YBA). We consider a set of lines of different types . A vector space $\mathcal{V}_I$ ($I \in \mathcal{J}$) is associated to each type of line. These lines may intersect and a YB operator is associated to each intersection



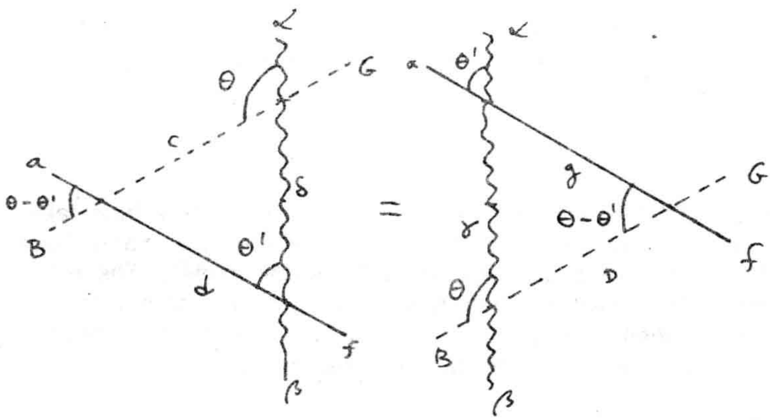
$$= \quad [T_{ab}(\mathcal{A},\mathcal{V})(\theta)]_{\alpha\beta} \qquad (1)$$

Here the lines ——— and ~~~~~~~~ are associated to the vector spaces $\mathcal{A}$ and $\mathcal{V}$ respectively. T(θ) is an operator acting on this couple of spaces , that is on $\mathcal{A} \otimes \mathcal{V}$. The indices a, b ($1 \leq a,b \leq \dim\mathcal{A}$) and α,β ($1 \leq \alpha,\beta \leq \dim\mathcal{V}$) label the basis vectors (states) in $\mathcal{A}$ and $\mathcal{V}$ respectively. The complex variable θ is called spectral parameter and describes geometrically the angle between the lines.

The operators T(θ) fulfil a YBA when the following relation holds

$$T^{(K,I)}(\theta-\theta')\ T^{(K,J)}(\theta)\ T^{(I,J)}(\theta') = T^{(I,J)}(\theta')\ T^{(K,J)}(\theta)\ T^{(K,I)}(\theta-\theta') \qquad (2)$$

for any choice of vector spaces $\mathcal{V}_I$, $\mathcal{V}_J$ and $\mathcal{V}_K$ (I, J and K $\in \mathcal{J}$) and any value of θ , $\theta' \varepsilon$ **C.** Graphically eq.(2) reads



(3)

Here, we use the convention that one must sum over all states in internal lines. That is lines linking two intersections. The graphical meaning of the YBA is clear : one can push any line through the intersection of two others keeping the value of the expression invariant. This invariance holds provided all angles θ are kept fixed under the displacement of the lines. Eq.(4) or (5) are called usually Yang-Baxter equation (YBE). It has different physical meanings in different contexts (see below).

The "diagonal" generators $T^{(I,I)}(\Theta)$ are R-matrices. The R-matrix associated with the lowest dimensional V^I is the fundamental one. Usually, the whole set of $T^{(I,J)}(\Theta)$ can be constructed once the fundamental R-matrix is found. Therefore this fundamental R-matrix, solution of eq.(2) for the lowest dimensional space I = J = K, defines the YB algebra.

The YBA (4) takes a particularly simple form when two vector spaces are identical. That is $\mathcal{V}_K = \mathcal{V}_I = \mathcal{A}$ and $\mathcal{V}_J = \mathcal{V}$ in eq.(4). One finds

$$R(\theta-\theta')\,[\,T(\theta)\otimes T(\theta')\,] = [\,T(\theta')\otimes T(\theta)\,]\,R(\theta-\theta') \qquad (4)$$

where $R = T^{(\mathcal{A},\mathcal{A})}$ and $T = T^{(\mathcal{A},\mathcal{V})}$ and we used the tensor product notation $(A\otimes B)_{ab,cd} = A_{ac}\,B_{bd}$. In eq.(4) there is an operational product of the T's in the space $\mathcal{V}$. More explicitly , eq.(4) reads

$$R_{ab}{}^{kl}(\theta-\theta')\;T_{kc}(\theta)\;T_{ed}(\theta') = T_{am}(\theta')\;T_{bn}(\theta)\;R_{cd}{}^{mn}(\theta-\theta') \quad (5).$$

The R-matrix (with components $R_{ab}{}^{cd}(\Theta-\Theta')$) for the lowest dimensional vector space $\mathcal{V}_0$ defines the YB algebra. These c-numbers play the role of "structure constants" and the $T_{ab}(\Theta)$ that of generators of the YB algebra.It must be noticed that the YB algebras are not in general Lie algebras but Hopf algebras (see below and ref.[1]). In summary a YB algebra is a set of operators $T^{(I,J)}(\theta)$ acting on a couple of vector spaces (V^I, V^J). V^I (V^J) stands for the auxiliary space (quantum space = $\mathcal{V}$) for this case. This set of operators is such that the relation (2) holds for any choice (I,J,K) of vector spaces.

The necessary and sufficient conditions for a YB algebra to be invariant under a group $\mathcal{G}$ is[2]

$$[\, R(\theta)\, ,\, g_{\mathcal{A}} \otimes g_{\mathcal{A}}\,] = 0 \quad , \quad \forall\, g \in \mathcal{G} \qquad (6)$$

Here $g_{\mathcal{A}}$ is the representation of $g \in \mathcal{G}$ acting on $\mathcal{A}$. We find from eqs.(5)-(6) that both $T(\theta)$ and $g_{\mathcal{A}}\, T(\theta)$ obey the same YBA. Therefore the YBA is invariant under $\mathcal{G}$.

This implies for the generators S of the Lie algebra $\mathcal{G}$ and the YB generators

$$[\, 1_{\mathcal{A}} \otimes S_{\mathcal{V}} + S_{\mathcal{A}} \otimes 1_{\mathcal{V}}\, ,\, T(\theta)\,] = 0 \qquad (7)$$

where $S_{\mathcal{A}}$ and $S_{\mathcal{V}}$ are the representations of S acting on the spaces $\mathcal{A}$ and $\mathcal{V}$ respectively. In eq.(7) $T(\theta)$ acts <u>both</u> on the spaces $\mathcal{A}$ and $\mathcal{V}$ as an operator.

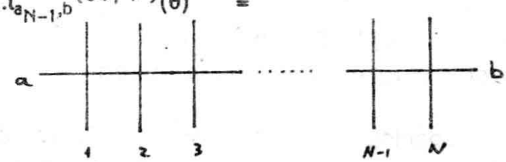
The YBA enjoy an invariance under shifts of the spectral parameter. Namely, if $T(\theta)$ fulfils the YBE (4), so does $T(\theta - \alpha)$ for fixed α

$$R(\theta-\theta')\, [\, T(\theta - \alpha) \otimes T(\theta' - \alpha)\,] = [\, T(\theta'-\alpha) \otimes T(\theta - \alpha)\,]\, R(\theta-\theta') \qquad (8)$$

The fundamental property of the YBA is the reproduction property. Namely, if $t^{(\mathcal{A},\mathcal{V})}(\theta)$ is a YB operator, one can build more general YB operators by the following rule

$$T_{ab}{}^{(\mathcal{A},\mathcal{V}^K)}(\theta) = \sum_{a_1,\ldots,a_{N-1}=1}^{\dim \mathcal{A}} t_{a,a_1}{}^{(\mathcal{A},\mathcal{V})}(\theta) \otimes t_{a_1,a_2}{}^{(\mathcal{A},\mathcal{V})}(\theta) \otimes \ldots\ldots$$

$$\ldots\ldots t_{a_{N-1},b}{}^{(\mathcal{A},\mathcal{V})}(\theta) \quad = \qquad (9)$$

=


After this short introduction to YB algebras (for details see ref.[1]), let us explain how massive relativistic QFT follow from YB algebras in appropiate scaling limits.

In two dimensions, the light-cone approach is the more general and precise way to construct integrable QFT and conformal invariant theories. One starts from integrable lattice models like vertex models on a diagonal lattice. This diagonal lattice is a discretization of Minkowski space-time in light-cone coordinates $X_{\pm} = X \pm T$. The matrix elements

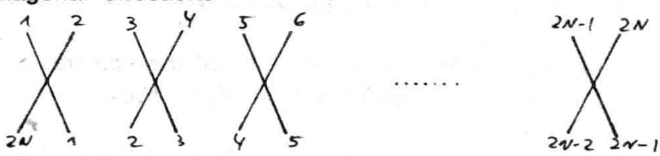
$$[\, T_{ab}(\theta)\,]_{\alpha}{}^{\beta} \quad (1 \leq \alpha\, ,\beta \leq \dim \mathcal{V}) \tag{10}$$

are now interpreted as quantum mechanical transition amplitudes for bare particles propagating to the right or to the left by the bonds at the speed of light. In the simplest case $\mathcal{A} = \mathcal{V}$, q = 2 and we interpret these particles as bare fermions without internal degrees of freedom. The allowed microscopical amplitudes, assuming a U(1) charge conservation, correspond to the six-vertex model weights in the statistical mechanical language. We can organize the microscopic amplitudes at a site into a unitary bare scattering matrix:

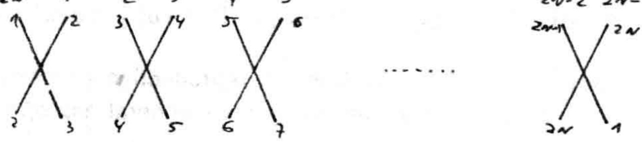
$R_{ab}{}^{cd}(\theta) =$ 

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix} \tag{11}$$

We can build now the operators describing the evolution by one lattice step in the diagonal directions

$U_R =$ 

(12)

$U_L =$ 

(13)

where the numbers 1, 2, 3,...,2N label the sites.

In this way the massive Thirring model is constructed both at the bare and at the renormalizerd levels [3]. The appropiate scaling limits are defined as $a \to 0$, $i\theta \to \infty$.

$$\text{Bare}: \sin\gamma \, \exp[-\theta\,] \,/\, a = \text{fixed.}$$
$$\text{Renormalized}: \sin\gamma \, \exp[-K\theta\,] \,/\, a = \text{fixed} \tag{14}$$

Here a is the lattice spacing , $K = \pi / \gamma$ and γ is the anisotropy parameter. The light-cone transfer matrices rigorously define the Hamiltonian and momentum as

$$H \pm P = (2i/a)\, \mathrm{Log}\, U_{R_L}(\theta) \qquad (15)$$

Using U_R and U_L , the Heisenberg equations of motion are derived in the lattice and their scaling limit obtained[3]. They lead to the MTM bare equations providing a precise identification of the fields, the coupling constant and the mass[3] .The exact spectrum of U_R and U_L follows from the usual row-to-row transfer matrices and provides the particle spectrum of the theory in the renormalized scaling limit (eq. (14)).

Let us say two words about the light cone transfer matrices U_R and U_L. The row-to-row inhomogeneous YB generators can be written as [1]

$$T_{ab}(\theta, \underset{\sim}{\alpha}) = \sum_{a_1,\dots,a_{N-1}}^{q} t_{aa_1}(\theta-\alpha_1) \otimes t_{a_1a_2}(\theta-\alpha_2) \otimes \dots t_{a_{N-1}b}(\theta-\alpha_N)$$
$$q = \dim \mathcal{A} \qquad (16)$$

where $[\, t_{ab}(\theta)\,]_{cd} = R_{ca}^{\ bd}(\theta)$ fullfils eq. (2).Eq.(16) also verifies the YB equation (2). For N = 2 this defines a coproduct of YB generators providing a Hopf algebra structure. This is a non commutative and non cocommutative Hopf algebra having in addition an antipode [1].

The row-to-row transfer matrix $\tau[\theta] = \sum_a T_{aa}[\theta]$ fulfills

$$[\tau[\theta, \underset{\sim}{\alpha}], \tau[\theta', \underset{\sim}{\alpha}]] = 0$$

thanks to eq.(4).In particular, if we set $\alpha_j = (-1)^{j+1}\theta$ [3].

$$\tau(\theta, \theta) = U_L(\theta) \quad , \quad \tau(-\theta, \theta) = U_R(\theta)^{\dagger} \qquad (17)$$

Therefore, the eigenvalue problem for U_R and U_L reduces to the one for $\tau(\theta, \underset{\sim}{\alpha})$. This problem is exactly solved by the Bethe Ansatz. That is, calling $B(\theta) \equiv T_{12}(\theta)$, the eigenvectors read [1]

$$\Psi_r(\theta_1, \theta_2, \dots, \theta_r) = B(\theta_1)\, B(\theta_2) \dots B(\theta_r)\, \Omega \qquad (18)$$

where Ω is the bare (ferroelectric) ground state;

$$\Omega = \binom{1}{0} \otimes \binom{1}{0} \otimes \dots\dots \otimes \binom{1}{0}$$

and the complex numbers θ_i fulfil the set of equations (Bethe Ansatz eqs.)

$$\left[\frac{\mathrm{sh}(\lambda_j + i\gamma/2)}{\mathrm{sh}(\lambda_j - i\gamma/2)}\right]^N = -\prod_{k=1}^{r} \frac{\mathrm{sh}(\lambda_j - \lambda_k + i\gamma)}{\mathrm{sh}(\lambda_j - \lambda_k - i\gamma)} \qquad (19)$$

Here $\lambda_j = -i(\theta_j - \gamma/2)$. These ψ_r are eigenvectors of S_z and $\tau(\theta)$ with eigenvalues N/2 - r and $\Lambda(\theta) = \Lambda_+(\theta) + \Lambda_-(\theta)$ respectively, where

$$\Lambda_+ = a(\theta)^N \prod_{j=1}^{r} \frac{a(\theta_j - \theta)}{b(\theta_j - \theta)} \quad , \quad \Lambda_- = b(\theta)^N \prod_{j=1}^{r} \frac{a(\theta - \theta_j)}{b(\theta - \theta_j)} \qquad (20)$$

This is the solution for the six vertex model.Eqs.(19) are easily solvable in the $N = \infty$ limit by Fourier transform. A method to solve them for finite size has been given in refs.[6].
The lattice light-cone approach also works for Chiral fermionic models with Lagrangian

$$\mathscr{L} = i\Psi \partial\!\!\!/ \Psi - g(\Psi \gamma_\mu t^\alpha \Psi)(\Psi \gamma^\mu t^\beta \Psi) K_{\alpha\beta} \qquad (21)$$

where Ψ transforms under an irreducible representation of the symmetry group $\mathfrak{G}$, t_α are the generators of $\mathfrak{G}$ and $K_{\alpha\beta}$ the Killing form.The equations of motion for the vector current (conservation and flatness) were derived for these models as well as their exact particle spectrum in ref. [3]. The renormalized scaling limit is here $\theta \to \infty$, $a \to 0$ with

$$\mu = \exp(-K\theta) / a \quad , \text{ fixed} \qquad (22)$$

where K is $2\pi/\gamma$ times the square length of the shortest simple root of $\mathfrak{G}$ in the normalization where $B(E_\alpha, E_{-\alpha}) = -1$. This construction extends to higher dimensional representations of $\mathfrak{G}$ yielding the principal chiral model in the limit of infinite dimension when $\mathfrak{G} = SU(N)$[4].

Relativistic QFT arise from all known gapless integrable models. That is from YB algebras realizations involving trigonometric functions of θ. Hyperbolic and elliptic YB algebras realizations do not lead to relativistic QFT[5].

A second scaling limit letting the lattice spacing $a \to 0$ with fixed θ yields conformal invariant models provided one starts from a gapless lattice model.That is, a rational or a trigonometric YB algebra. The finite size resolution methods for the Bethe Ansatz (BA) equations of the lattice models [6] give the values of the central charge c and the conformal weights $(\Delta, \bar{\Delta})$ of operators for a large class of models [6,7].Branching coefficients can be related to one point functions[8].

Let us briefly summarize the results. In all cases , the dominant finite size corrections to the n^{th} eigenvalue of the vertex model transfer matrix $\tau(\theta)$, computed from the BA have the form