MATHEMATICS

The Loss of Certainty

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Mathematics

To my wife

Helen Mann Kline

Preface

This book treats the fundamental changes that man has been forced to make in his understanding of the nature and role of mathematics. We know today that mathematics does not possess the qualities that in the past earned for it universal respect and admiration. Mathematics was regarded as the acme of exact reasoning, a body of truths in itself, and the truth about the design of nature. How man came to the realization that these values are false and just what our present understanding is constitute the major themes. A brief statement of these themes is presented in the Introduction. Some of the material could be gleaned from a detailed technical history of mathematics. But those people who are interested primarily in the dramatic changes that have taken place will find that a direct, non-technical approach makes them more readily accessible and more intelligible.

Many mathematicians would perhaps prefer to limit the disclosure of the present status of mathematics to members of the family. To air these troubles in public may appear to be in bad taste, as bad as airing one's marital difficulties. But intellectually oriented people must be fully aware of the powers of the tools at their disposal. Recognition of the limitations, as well as the capabilities, of reason is far more beneficial than blind trust, which can lead to false ideologies and even to destruction.

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The gods have not revealed all things from the beginning,
But men seek and so find out better in time.

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Let us suppose these things are like the truth.

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But surely no man knows or ever will know
The truth about the gods and all I speak of.
For even if he happens to tell the perfect truth,
He does not know it, but appearance is fashioned over everything.

XENOPHANES

Introduction: The Thesis

To foresee the future of mathematics, the true method is to study its history and its present state. HENRI POINCARÉ

There are tragedies caused by war, famine, and pestilence. But there are also intellectual tragedies caused by limitations of the human mind. This book relates the calamities that have befallen man's most effective and unparalleled accomplishment, his most persistent and profound effort to utilize human reason—mathematics.

Put in other terms, this book treats on a non-technical level the rise and decline of the majesty of mathematics. In view of its present immense scope, the increasing, even flourishing, mathematical activity, the thousands of research papers published each year, the rapidly growing interest in computers, and the expanded search for quantitative relationships especially in the social and biological sciences, how can we talk about the decline of mathematics? Wherein lies the tragedy? To answer these questions we must consider first what values won for mathematics its immense prestige, respect, and glory.

From the very birth of mathematics as an independent body of knowledge, fathered by the classical Greeks, and for a period of over two thousand years, mathematicians pursued truth. Their accomplishments were magnificent. The vast body of theorems about number and geometric figures offered in itself what appeared to be an almost endless vista of certainty.

Beyond the realm of mathematics proper, mathematical concepts and derivations supplied the essence of remarkable scientific theories. Though the knowledge obtained through the collaboration of mathematics and science employed physical principles, these seemed to be as secure as the principles of mathematics proper because the predictions in the mathematical theories of astronomy, mechanics, optics, and hydrodynamics were in remarkably accurate accord with observation and experiment. Mathematics, then, provided a firm grip on the workings of nature, an understanding which dissolved mystery and replaced it by law and order. Man could pridefully survey the world about him and boast that he had grasped many of the secrets of the universe, which in essence were a series of mathematical laws. The conviction that mathematicians were securing truths is epitomized in Laplace's remark that Newton was a most fortunate man because there is just one universe and Newton had discovered its laws.

To achieve its marvelous and powerful results, mathematics relied upon a special method, namely, deductive proof from self-evident principles called axioms, the methodology we still learn, usually in high school geometry. Deductive reasoning, by its very nature, guarantees the truth of what is deduced if the axioms are truths. By utilizing this seemingly clear, infallible, and impeccable logic, mathematicians produced apparently indubitable and irrefutable conclusions. This feature of mathematics is still cited today. Whenever someone wants an example of certitude and exactness of reasoning, he appeals to mathematics.

The successes mathematics achieved with its methodology attracted the greatest intellectuals. Mathematics had demonstrated the capacities, resources, and strengths of human reason. Why should not this methodology be employed, they asked, to secure truths in fields dominated by authority, custom, and habit, fields such as philosophy, theology, ethics, aesthetics, and the social sciences? Man's reason, so evidently effective in mathematics and mathematical physics, could surely be the arbiter of thought and action in these other fields and obtain for them the beauty of truths and the truths of beauty. And so, during the period called the Enlightenment or the Age of Reason, mathematical methodology and even some mathematical concepts and theorems were applied to human affairs.

The most fertile source of insight is hindsight. Creations of the early 19th century, strange geometries and strange algebras, forced mathematicians, reluctantly and grudgingly, to realize that mathematics proper and the mathematical laws of science were not truths. They found, for example, that several differing geometries fit spatial experience equally well. All could not be truths. Apparently mathematical design was not inherent in nature, or if it was, man's mathematics was not necessarily the account of that design. The key to reality had been lost. This realization was the first of the calamities to befall mathematics.

The creation of these new geometries and algebras caused mathematicians to experience a shock of another nature. The conviction that they were obtaining truths had entranced them so much that they had rushed impetuously to secure these seeming truths at the cost of sound reasoning. The realization that mathematics was not a body of truths shook their confidence in what they had created, and they undertook to reexamine their creations. They were dismayed to find that the logic of mathematics was in sad shape.

In fact mathematics had developed illogically. Its illogical development contained not only false proofs, slips in reasoning, and inadvertent mistakes which with more care could have been avoided. Such blunders there were aplenty. The illogical development also involved inadequate understanding of concepts, a failure to recognize all the principles of logic required, and an inadequate rigor of proof; that is, intuition, physical arguments, and appeal to geometrical diagrams had taken the place of logical arguments.

However, mathematics was still an effective description of nature. And mathematics itself was certainly an attractive body of knowledge and in the minds of many, the Platonists especially, a part of reality to be prized in and for itself. Hence mathematicians decided to supply the missing logical structure and to rebuild the defective portions. During the latter half of the 19th century the movement often described as the rigorization of mathematics became the outstanding activity.

By 1900 the mathematicians believed they had achieved their goal. Though they had to be content with mathematics as an approximate description of nature and many even abandoned the belief in the mathematical design of nature, they did gloat over their reconstruction of the logical structure of mathematics. But before they had finished toasting their presumed success, contradictions were discovered in the reconstructed mathematics. Commonly these contradictions were referred to as paradoxes, a euphemism that avoids facing the fact that contradictions vitiate the logic of mathematics.

The resolution of the contradictions was undertaken almost immediately by the leading mathematicians and philosophers of the times. In effect four different approaches to mathematics were conceived, formulated, and advanced, each of which gathered many adherents. These foundational schools all attempted not only to resolve the known contradictions but to ensure that no new ones could ever arise, that is, to establish the consistency of mathematics. Other issues arose in the foundational efforts. The acceptability of some axioms and some principles of deductive logic also became bones of contention on which the several schools took differing positions.

As late as 1930 a mathematician might perhaps have been content with accepting one or another of the several foundations of mathematics and declared that his mathematical proofs were at least in accord with the tenets of that school. But disaster struck again in the form of a famous paper by Kurt Gödel in which he proved, among other significant and disturbing results, that the logical principles accepted by the several schools could not prove the consistency of mathematics. This, Gödel showed, cannot be done without involving logical principles so dubious as to question what is accomplished. Gödel's theorems produced a debacle. Subsequent developments brought further complications. For example, even the axiomatic-deductive method so highly regarded in the past as *the* approach to exact knowledge was seen to be flawed. The net effect of these newer developments was to add to the variety of possible approaches to mathematics and to divide mathematicians into an even greater number of differing factions.

The current predicament of mathematics is that there is not one but many mathematics and that for numerous reasons each fails to satisfy the members of the opposing schools. It is now apparent that the concept of a universally accepted, infallible body of reasoning—the majestic mathematics of 1800 and the pride of man—is a grand illusion. Uncertainty and doubt concerning the future of mathematics have replaced the certainties and complacency of the past. The disagreements about the foundations of the "most certain" science are both surprising and, to put it mildly, disconcerting. The present state of mathematics is a mockery of the hitherto deep-rooted and widely reputed truth and logical perfection of mathematics.

There are mathematicians who believe that the differing views on what can be accepted as sound mathematics will some day be reconciled. Prominent among these is a group of leading French mathematicians who write under the pseudonym of Nicholas Bourbaki:

> Since the earliest times, all critical revisions of the principles of mathematics as a whole, or of any branch of it, have almost invariably followed periods of uncertainty, where contradictions did appear and had to be resolved. . . . There are now twenty-five centuries during which the mathematicians have had the practice of correcting their errors and thereby seeing their science enriched, not impoverished; this gives them the right to view the future with serenity.

However, many more mathematicians are pessimistic. Hermann Weyl, one of the greatest mathematicians of this century, said in 1944:

> The question of the foundations and the ultimate meaning of mathematics remains open; we do not know in what direction it will find its final solution or even whether a final objective answer can be expected at all. "Mathematizing" may well be a creative activity of man, like language or music, of primary originality, whose historical decisions defy complete objective rationalization.

In the words of Goethe, "The history of a science is the science itself."

The disagreements concerning what correct mathematics is and the variety of differing foundations affect seriously not only mathematics proper but most vitally physical science. As we shall see, the most well-developed physical theories are entirely mathematical. (To be sure, the conclusions of such theories are interpreted in sensuous or truly physical objects, and we hear voices over our radios even though we have not the slightest physical understanding of what a radio wave is.) Hence scientists, who do not personally work on foundational problems, must nevertheless be concerned about what mathematics can be confidently employed if they are not to waste years on unsound mathematics.

The loss of truth, the constantly increasing complexity of mathematics and science, and the uncertainty about which approach to mathematics is secure have caused most mathematicians to abandon science. With a "plague on all your houses" they have retreated to specialties in areas of mathematics where the methods of proof seem to be safe. They also find problems concocted by humans more appealing and manageable than those posed by nature.

The crises and conflicts over what sound mathematics is have also discouraged the application of mathematical methodology to many areas of our culture such as philosophy, political science, ethics, and aesthetics. The hope of finding objective, infallible laws and standards has faded. The Age of Reason is gone.

Despite the unsatisfactory state of mathematics, the variety of approaches, the disagreements on acceptable axioms, and the danger that new contradictions, if discovered, would invalidate a great deal of mathematics, some mathematicians are still applying mathematics to physical phenomena and indeed extending the applied fields to economics, biology, and sociology. The continuing effectiveness of mathematics suggests two themes. The first is that effectiveness can be used as the criterion of correctness. Of course such a criterion is provisional. What is considered correct today may prove wrong in the next application.

The second theme deals with a mystery. In view of the disagreements about what sound mathematics is, why is it effective at all? Are we performing miracles with imperfect tools? If man has been deceived, can nature also be deceived into yielding to man's mathematical dictates? Clearly not. Yet, do not our successful voyages to the moon and our explorations of Mars and Jupiter, made possible by technology which itself depends heavily on mathematics, confirm mathematical theories of the cosmos? How can we, then, speak of the artificiality and varieties of mathematics? Can the body live on when the mind and spirit are bewildered? Certainly this is true of human beings and it is true of mathematics. It behooves us therefore to learn why, despite its uncertain foundations and despite the conflicting theories of mathematicians, mathematics has proved to be so incredibly effective.

I

The Genesis of Mathematical Truths

> Thrice happy souls! to whom 'twas given to rise
> To truths like these, and scale the spangled skies!
> Far distant stars to clearest view they brought,
> And girdled ether with their chains of thought.
> So heaven is reached—not as of old they tried
> By mountains piled on mountains in their pride.
>
> OVID

Any civilization worthy of the appelation has sought truths. Thoughtful people cannot but try to understand the variety of natural phenomena, to solve the mystery of how human beings came to dwell on this earth, to discern what purpose life should serve, and to discover human destiny. In all early civilizations but one, the answers to these questions were given by religious leaders, answers that were generally accepted. The ancient Greek civilization is the exception. What the Greeks discovered—the greatest discovery made by man—is the power of reason. It was the Greeks of the classical period, which was at its height during the years from 600 to 300 B.C., who recognized that man has an intellect, a mind which, aided occasionally by observation or experimentation, can discover truths.

What led the Greeks to this discovery is a question not readily answered. The initiators of the plan to apply reason to human affairs and concerns lived in Ionia, a Greek settlement in Asia Minor, and many historians have sought to account for the happenings there on the basis of political and social conditions. For example, the Ionians were rather freer to disregard the religious beliefs that dominated the European Greek culture. However, our knowledge of Greek history before about 600 B.C. is so fragmentary that no definitive explanation is available.

In the course of time the Greeks applied reason to political systems, ethics, justice, education, and numerous other concerns of man. Their chief contribution, and the one which decisively influenced all later cultures, was to undertake the most imposing challenge facing reason, learning the laws of nature. Before the Greeks made this contribution, they and the other civilizations of ancient times regarded nature as chaotic, capricious, and even terrifying. Acts of nature were either unexplained or attributed to the arbitrary will of gods who could be propitiated only by prayers, sacrifices, and other rituals. The Babylonians and Egyptians, who had notable civilizations as far back as 3000 B.C., did note some periodicities in the motions of the sun and moon and indeed based their calendars on these periodicities but saw no deeper significance in them. These few exceptional observations did not influence their attitude toward nature.

The Greeks dared to look nature in the face. Their intellectual leaders, if not the people at large, rejected traditional doctrines, supernatural forces, superstitions, dogma, and other trammels on thought. They were the first people to examine the multifarious, mysterious, and complex operations of nature and to attempt to understand them. They pitted their minds against the welter of seemingly haphazard occurrences in the universe and undertook to throw the light of reason upon them.

Possessed of insatiable curiosity and courage, they asked and answered the questions that occur to many, are tackled by few, and are resolved only by individuals of the highest intellectual caliber. Is there any plan underlying the workings of the entire universe? Are plants, animals, men, planets, light, and sound mere physical accidents or are they part of a grand design? Because they were dreamers enough to arrive at new points of view, the Greeks fashioned a conception of the universe which has dominated all subsequent Western thought.

The Greek intellectuals adopted a totally new attitude toward nature. This attitude was rational, critical, and secular. Mythology was discarded as was the belief that the gods manipulate man and the physical world according to their whims. The intellectuals eventually arrived at the doctrine that nature is orderly and functions invariably according to a grand design. All phenomena apparent to the senses, from the motions of the planets to the stirrings of the leaves on a tree, can be fitted into a precise, coherent, intelligible pattern. In short, nature is rationally designed and that design, though unaffected by human actions, can be apprehended by man's mind.

The Greeks were not only the first people with the audacity to conceive of law and order in the welter of phenomena but also the first with the genius to uncover some of the underlying patterns to which nature apparently conforms. Thus they dared to ask for, and found, design underlying the greatest spectacle man beholds, the motion of the brilliant sun, the changing phases of the many hued moon, the