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Trygve Johnsen
Andreas Leopold Knutsen

*K*3 Projective Models in Scrolls

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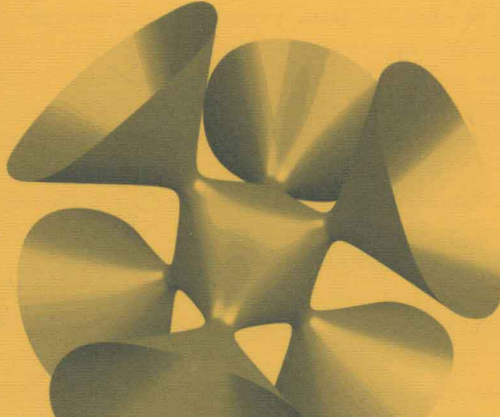

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Preface

The cover picture shows a smooth quartic surface in space, the simplest example of a projective model of a $K3$ surface. In the following pages we will encounter many more examples of models of such surfaces.

The purpose of this volume is to study and classify projective models of complex $K3$ surfaces polarized by a line bundle L such that all smooth curves in $|L|$ have non-general Clifford index. Such models are in a natural way contained in rational normal scrolls.

These models are *special* in moduli in the sense that they do not represent the general member in the countable union of 19-dimensional families of polarized $K3$ surfaces. However, they are of interest because they fill up the set of models in $\mathbf{P}^g$ for $g \leq 10$ not described as complete intersections in projective space or in a homogeneous space as described by Mukai, with a few classificable exceptions.

Thus our study enables us to classify and describe all projective models of $K3$ surfaces of genus $g \leq 10$, which is the main aim of the volume.

Acknowledgements. We thank Kristian Ranestad, who suggested to study certain projective models of $K3$ surfaces in scrolls that had shown up in connection with his work on varieties of sums of powers (see [I-R1], [I-R2] and [R-S]). This idea was the starting point of our work.

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Bergen/Oslo, March 2004

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1

Introduction

1.1 Background

A $K3$ surface is a smooth compact complex connected surface with trivial canonical bundle and vanishing first Betti number. The mysterious name $K3$ is explained by A. Weil in the comment on his *Final report on contract AF18(603)-57* (see [We] p 546):

> *Dans la seconde partie de mon rapport, il s'agit des variétés kählerïennes dites K3, ainsi nommées en l'honneur de Kummer, Kodaira, Kähler et de la belle montagne K2 au Cachemire.*

It is well known that all $K3$ surfaces are diffeomorphic, and that there is a 20-dimensional family of analytical isomorphism classes of $K3$ surfaces. However, the general element in this family is not algebraic, in fact the algebraic ones form a countable union of 19-dimensional families. More precisely, for any $n > 0$ there is a 19-dimensional irreducible family of $K3$ surfaces equipped with a base point free line bundle of self-intersection n. Moreover, the family of $K3$ surfaces having $\geq k$ linearly independent divisors (i.e. the surfaces with *Picard number* $\geq k$, where the Picard number is by definition the rank of the Picard group) forms a dense countable union of subvarieties of dimension $20 - k$ in the family of all $K3$ surfaces. In particular, on the general algebraic $K3$ surface all divisors are linearly equivalent to some rational multiple of the hyperplane class (see [G-H, pp. 590–594]).

A pair (S, L) of a $K3$ surface S and a base point free line bundle L with $L^2 = 2g - 2$ will be called a *polarized* $K3$ surface of genus g. Note that $g = h^0(L) - 1$ and that g is the arithmetic genus of any member of $|L|$. The sections of L give a map φ_L of S to $\mathbf{P}^g$, and the image is called a projective model of S. When φ_L is birational, the image is a surface of degree $2g-2$ in $\mathbf{P}^g$. It is also easy to see that a projective model of genus 2 is a $2:1$ map $S \to \mathbf{P}^2$ branched along a sextic curve.

A very central point in the theory of projective models of $K3$ surfaces is that by the adjunction formula every smooth hyperplane section of a projective model of S (these are the images by φ_L of the smooth members of $|L|$) are canonical curves, i.e. curves for which $\omega_C \simeq \mathcal{O}_C(1)$.

The first examples of projective models of $K3$ surfaces are the ones which are complete intersections in projective space. Using the fact that for a complete intersection surface S of $n-2$ hypersurfaces in $\mathbf{P}^n$ of degrees $d_1, \ldots, d_{n-2}$ we have $\omega_S \simeq \mathcal{O}_S(\sum d_i - n - 1)$ and $h^1(\mathcal{O}_S) = 0$ (see e.g. [Hrts, Exercises II,8.4 and III,5,5]), we find that there are exactly three types of $K3$ complete intersections, namely a hyperquartic in $\mathbf{P}^3$, a complete intersection of a hyperquadric and a hypercubic in $\mathbf{P}^4$ and a complete intersection of three hyperquadrics in $\mathbf{P}^5$.

In fact one can show that any birational projective model of genus 3 is a quartic surface and of genus 4 a complete intersection of a quadric and a cubic hypersurface. But already for genus 5 the situation is not as simple: The general model is a complete intersection of three hyperquadrics, but there are models which are not. In fact, take a 3-dimensional smooth rational normal scroll X of degree 3 in $\mathbf{P}^5$, which can be seen as the union of $\mathbf{P}^2$s parametrized by $\mathbf{P}^1$, i.e. a $\mathbf{P}^2$-bundle over $\mathbf{P}^1$. Intersect this scroll by a (sufficiently general) cubic hypersurface $\mathcal{C}$ containing one of the $\mathbf{P}^2$-fibers, call it F, then the intersection is $\mathcal{C} \cap X = F \cup S$, where S is a smooth surface of degree 8 in $\mathbf{P}^5$, i.e. a $K3$ surface. The ideal of this surface cannot be generated only by quadrics, whence S is not a complete intersection of three hyperquadrics. Note that the intersection of $\mathcal{C}$ with a general $\mathbf{P}^2$-fiber is a smooth curve of degree 3 in $\mathbf{P}^2$, which is elliptic by the genus formula, so S contains a pencil of elliptic curves of degree 3. In particular, since such a curve cannot be linearly equivalent to a multiple of the hyperplane section, S contains two linearly independent divisors, whence these surfaces can at most fill up an 18-dimensional family (in fact we will show that they do fill up an 18-dimensional family). Another interesting point is that the elliptic pencil on the surface cuts out a g^1_3 (i.e. a linear system of dimension 1 and degree 3) on each hyperplane section of S. Conversely, by a classical theorem of Enriques-Petri, the homogeneous ideal of a canonical curve with a g^1_3 is generated by both quadrics and cubics, so any projective model in $\mathbf{P}^5$ of a $K3$ surface whose hyperplane sections have a g^1_3 cannot be the complete intersection of three hyperquadrics.

For $6 \leq g \leq 10$ and $g = 12$ it is shown by Mukai in [Mu1] and [Mu2] that the general projective models are complete intersections in certain homogeneous varieties contained in projective spaces of larger dimension than g. The ambient varieties are constructed using special divisors on the hyperplane sections, and the general models have the property that their hyperplane sections do not carry certain particular g^r_ds induced from divisors on the surface.

That the projective model of a $K3$ surface somehow has to do with special divisors carried by the curves in $|L|$ dates back to the classical paper [SD] of Saint-Donat, which has become the main reference for all later work on projective models of or curves on $K3$ surfaces.

As remarked in [SD] it is clear from Zariski's Main Theorem (see e.g. [Hrts, V, Thm. 5.2]) that

$$\varphi_L = u_L \circ \theta_L,$$

where u_L is a finite morphism and θ_L maps S birationally onto a normal surface by contracting finitely many curves to rational double points and is an isomorphism outside these curves (the contracted curves are the curves sent to a point, and these are precisely the curves Δ such that $L.\Delta = 0$).

One of the main results in [SD] describes exactly when the map u_L is an identity, in other words when φ_L is birational.

Theorem 1.1 (Saint-Donat [SD]). *Let L be a base point free line bundle with $L^2 > 0$ on a $K3$ surface S. The following conditions are equivalent:*

(a) φ_L is not birational.
(b) There is a smooth hyperelliptic curve in $|L|$.
(c) All the smooth curves in $|L|$ are hyperelliptic.
(d) $L^2 = 2$; or there is a smooth elliptic curve E on S satisfying $E.L = 2$; or $L \sim 2B$ for a smooth curve B with $B^2 = 2$ and $L \sim 2B$.

A linear system $|L|$ satisfying these properties is said to be hyperelliptic.

Furthermore, if L is not hyperelliptic, then the natural maps $S_n H^0(L) \to H^0(nL)$ are surjective for all n

(Recall that a smooth curve is said to be hyperelliptic if it carries a g^1_2.) This "lifts" the classical fact that the canonical morphism of a smooth curve is an embedding if and only if the curve is not hyperelliptic and also Noether's theorem, to the surface:

Theorem 1.2 (Noether [No]). *If C is not hyperelliptic, then the ring $\oplus H^0(C, n\omega_C)$ is the homogeneous coordinate ring of C in its canonical embedding in $\mathbf{P}^g$.*

Moreover, Saint-Donat's result tells that a g^1_2 on a smooth curve on a $K3$ surface "propagates" to the other smooth members of the linear system. In fact, except for the trivial case where all the curves have genus 2 (the case $L^2 = 2$) and are therefore trivially hyperelliptic, such a propagating g^1_2 is given by the pencils $\mathcal{O}_C(E)$ or $\frac{1}{2}\mathcal{O}_C(B)$ for any smooth curve $C \in |L|$, corresponding to the curves E and B in (d).

Another main result in [SD] describes the homogeneous ideal of the image $\varphi_L(S)$:

Theorem 1.3 (Saint-Donat [SD]). *Let L be a base point free non-hyperelliptic line bundle with $L^2 \geq 8$ on a $K3$ surface S. Denote by I the graded ideal defined as the kernel of the map $S_* H^0(L) \to \oplus H^0(nL)$. Then I is generated by quadrics and cubics. Moreover the following conditions are equivalent:*

(a) I is generated not only by quadrics.
(b) $|L|$ contains a smooth curve carrying a g^1_3 or a g^2_5.
(c) All the smooth curves in $|L|$ carry a g^1_3 or all carry a g^2_5.

(d) *There is a smooth elliptic curve* E *on* S *satisfying* $E.L = 3$; *or* $L \sim 2B + \Gamma$ *for a smooth curve* B *with* $B^2 = 2$ *and* Γ *a smooth rational curve with* $B.\Gamma = 1$ *(and* $\Gamma^2 = -2$*, in particular* $L^2 = 10$*).*

Again, this lifts the classical result of Petri from the curve to the surface:

Theorem 1.4 (Petri [Pe]). *The homogeneous ideal of a non-hyperelliptic canonical curve* C *is generated by quadrics, unless* C *has a* g^1_3 *or a* g^2_5*.*

In the cases $L^2 = 4$ or 6 all the smooth curves in $|L|$ have genus 3 or 4, so they necessarily carry a g^1_3 (i.e. they are *trigonal*). For higher genus the last result again tells that g^1_3s and g^2_5s "propagate" among the smooth curves in $|L|$. Indeed the linear systems $|E|$ and $|B|$ on S given in (d) cut out a g^1_3 and a g^2_5 respectively on all the members of $|L|$.

Moreover, Saint-Donat gives a thorough description of the projective models in the special cases where $|L|$ is hyperelliptic or I is generated not only by quadrics. The models happen to lie in rational normal scrolls.

To broaden our perspective, let us recall the definition of the *Clifford index* of a smooth curve C of genus g, introduced by H. H. Martens in [HMa]. This is denoted by Cliff C and is the minimal integer $\deg A - 2(h^0(A) - 1)$ for all line bundles A on C satisfying $h^0(A) \geq 2$ and $h^1(A) \geq 2$. (The latter requirements presuppose that $g \geq 4$; however one can give ad hoc definitions in the cases of genus 2 or 3, by setting Cliff $C = 0$ for C of genus 2 or hyperelliptic of genus 3, and Cliff $C = 1$ for C non-hyperelliptic of genus 3.) Clifford's theorem states that Cliff $C \geq 0$ with equality if and only if C is hyperelliptic and Cliff $C = 1$ if and only if C is trigonal or a smooth plane quintic. Moreover, we also have Cliff $C \leq \lfloor \frac{g-1}{2} \rfloor$, with equality for the general curve (cf. [A-C-G-H, V]).

We can rephrase the two results above of Saint-Donat by saying that φ_L is birational if and only if Cliff $C > 0$ for every smooth curve $C \in |L|$ and that in addition I is generated only by quadrics if and only if Cliff $C > 1$ for every smooth curve $C \in |L|$.

Moreover, Saint-Donat's results yield that either all or none of the smooth curves in a complete linear system on a $K3$ surface have Clifford index 0 (resp. 1). It is then a natural question to ask whether this also holds for higher indices.

Around ten years after the appearance of Saint-Donat's paper, interesting new techniques were introduced in the study of projective varieties.

One tool was the introduction of *Koszul cohomology* in [Gr] in connection with the study of *syzygies* and the resulting famous conjecture of Green.

Consider a smooth variety X with a base point free line bundle L with $r := h^0(L) - 1$ on it and the graded ring $R := \oplus_{m \geq 0} H^0(X, mL)$. This is in a natural way a finitely generated module over $T := \operatorname{Sym} H^0(X, L)$, the coordinate ring of the projective space $\mathbf{P}(H^0(L))$, and so has a *minimal graded free resolution*

$$0 \longrightarrow M_{r-1} \longrightarrow \ldots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow R \longrightarrow 0,$$

where each M_i is a direct sum of twists of T:

$$M_i = \oplus_j T(-j) \otimes M_{i,j} \simeq \oplus_j T(-j)^{\beta_{i,j}}.$$

The finite dimensional vector space $M_{i,j}$ is called the *syzygy* of order i and weigth j and the $\beta_{i,j} := \dim M_{i,j}$ are called the *graded Betti-numbers.* Now L is said to satisfy property N_p if

$$M_0 = T \quad \text{and} \quad M_i = T(-i-1)^{\beta_{i,i-1}} \quad \text{for all} \quad 1 \leq i \leq p.$$

To be more concrete, N_0 means that $\varphi_L(X)$ is projectively normal, N_1 that in addition its homogeneous ideal is generated by quadrics, and more generally N_p for $p \geq 2$ means that in addition the matrices in the minimal graded free resolution have linear entries from the second to the pth step.

Now if $X = C$ is a smooth curve Green conjectured the following:

Conjecture 1.5 (Green [Gr]). *The Clifford index of C is the least integer p for which property N_p fails for the canonical bundle.*

For Cliff $C = 0$ this is Noether's theorem and for Cliff $C = 1$ this is Petri's theorem.

A "Lefschetz theorem" as in [Gr, (3.b.7)] implies that the syzygies of a hyperplane section of a $K3$ surface are the same as the ones of the $K3$ surface, so that all linearly equivalent smooth curves on a $K3$ surface have the same syzygies. Therefore an immediate consequence of Green's conjecture would be that all the smooth curves in a linear system on a $K3$ surface have the same Clifford index (since all such are canonically embedded by φ_L, by the adjunction formula).

A second important tool was the vector bundle techniques introduced by Lazarsfeld [La2] and Tyurin [Ty] (and also by Reider [Rdr] in a slightly different context). Using these techniques Green and Lazarsfeld [G-L4] proved that all the smooth curves in a linear system on a $K3$ surface have the same Clifford index. Moreover, they proved that if non-general, i.e. if $< \lfloor \frac{g-1}{2} \rfloor$, the Clifford index is induced by a line bundle on the surface, similarly to the cases studied by Saint-Donat.

Theorem 1.6 (Green-Lazarsfeld [G-L4]). *Let L be a base point free line bundle on a $K3$ surface S with $L^2 > 0$. Then Cliff C is constant for all smooth irreducible $C \in |L|$, and if Cliff $C < \lfloor \frac{g-1}{2} \rfloor$, then there exists a line bundle M on S such that $M_C := M \otimes \mathcal{O}_C$ computes the Clifford index of C for all smooth irreducible $C \in |L|$.*

As an immediate consequence we see that in the general case, i.e. when Pic $S \simeq \mathbf{Z}L$, then there can exist no line bundle M as above, so on the general $K3$ surface all curves have the general Clifford index.

By the result of Green and Lazarsfeld it makes sense to define the Clifford index Cliff L of a base point free line bundle, or the Clifford index Cliff $_L(S)$

of a polarized $K3$ surface (S, L), as the Clifford index of the smooth curves in $|L|$.

The fact that the Clifford index somehow influences the projective model of S was also remarked in [Kn4], where the second author studies higher order embeddings of $K3$ surfaces. Roughly speaking the Clifford index determines the amount of $(k+1)$-secant $(k-1)$-planes of the projective model.

In this book we study the projective models of those polarized $K3$ surfaces of genus g of *non-general Clifford index*, i.e. with Cliff $_L(S) < \lfloor \frac{g-1}{2} \rfloor$. These surfaces are special in moduli, since they can only fill up at most 18-dimensional families (except in the particular cases where S has Picard number one and L is non-primitive, i.e. L is an integral multiple ≥ 2 of the generator of Pic S).

As in the cases of Clifford index 0 and 1 studied by Saint-Donat, these models lie in rational normal scrolls in a natural way.

The central point is that by the result of Green and Lazarsfeld there exists in these cases a linear system $|D|$ on S computing the Clifford index of L. We can moreover choose such a linear system which is base point free and such that the general member is a smooth curve. We call such a divisor (class) D a *free Clifford divisor* for L.

The images of the members of $|D|$ by φ_L span sublinear spaces inside $\mathbf{P}^g$. Each subpencil $\{D_\lambda\}$ within the complete linear system $|D|$ then gives rise to a pencil of sublinear spaces. For each fixed pencil the union of these spaces will be a rational normal scroll $\mathcal{T}$. These scrolls are the natural ambient spaces for non-Clifford general $K3$ surfaces. Our description is inspired by and uses methods developed by Schreyer in [Sc], where the authour studies scrolls containing canonical curves and uses this to prove Green's conjecture for $g \leq 8$. In the same spirit as Saint-Donat, we so to speak lift Schreyer's results from the curve to the surface.

In the cases of Clifford index 1 and 2 with $D^2 = 0$, the description of the projective models is particularly nice, since they are then complete intersections in their corresponding scrolls.

Another important tool, which was still not available at the time [SD] was written, are the results on lattices by Nikulin [Ni], which allows to construct families of $K3$ surfaces with prescribed lattices, and thus show the existence of several interesting families. Using this, the second author proved the following *Existence Theorem* in [Kn2]: For any pair of integers (g, c) such that $g \geq 2$ and $0 \leq c \leq \lfloor \frac{g-1}{2} \rfloor$, there exists an 18-dimensional family of polarized $K3$ surfaces of genus g and of Clifford index c. Similar techniques allow us to prove the existence of all the families we study in this book and also compute their number of moduli.

We also give a description of those projective models for $g \leq 10$ that are Clifford general, but still not general in the sense of Mukai (i.e. they are not complete intersections in homogeneous spaces). These models are also contained in scrolls, and can be analysed in a similar manner. Together with Mukai's results this then gives a complete picture of the birational projective models for $g \leq 10$. For $g = 11$ and $g \geq 13$ our description of non-Clifford general projective models is not supplemented by any description of general projective models at all. We hope, however, that our description of the non-general models may have some interest in themselves.

1.2 Related literature

$K3$ surfaces in scrolls have also been studied in [Br] and [Ste].

Saint-Donat's results on the propagation of g^1_2s and g^1_3s among the smooth curves in a linear system on a $K3$ surface were extended to other g^1_ds by Reid [Re3]. The general question of propagation of g^r_ds came out of work of Harris and Mumford [H-M]. In fact they conjectured (unpublished) that the gonality (i.e. the minimal degree of a pencil on a curve) should be constant among the smooth curves in a linear system. Subsequently, Donagi and Morrison [D-M] pointed out the following counterexample:

Example 1.7. [D-M, (2.2)] Let $\pi : S \to \mathbf{P}^2$ be a $K3$ surface of genus 2, i.e. a double cover of $\mathbf{P}^2$ branched along a smooth sextic, and let $L := \pi^* \mathcal{O}_{\mathbf{P}^2}(3)$. The arithmetic genus of the curves in $|L|$ is 10. We have $H^0(L) = \pi^* H^0 \mathcal{O}_{\mathbf{P}^2}(3) \oplus W$, where W is the one-dimensional subspace of sections vanishing on the ramification locus. The smooth curves C in the first summand are double covers of cubics, whence tetragonal (they all carry a 1-parameter family of g^1_4s which is the pullback of the 1-parameter family of g^1_2s on $\pi(C)$). On the other hand, the general curve in $|L|$ is isomorphic to a smooth plane sextic and is therefore of gonality 5. (Note that, in full accordance with the theorem of Green and Lazarsfeld, all the curves have Clifford index 1.)

The question is still open whether there exist other counterexamples. Ciliberto and Pareschi [C-P] proved that this is indeed the only counterexample when L is ample.

Exceptional curves, i.e. curves for which the Clifford index is not computed by a pencil, so that Cliff $C <$ gon $C - 2$, were studied in [E-L-M-S], where a whole class of examples were constructed as curves on $K3$ surfaces.

As for other surfaces, the constancy of the Clifford index and gonality of the smooth curves in a linear system on a Del Pezzo surface was studied by Pareschi [Pa] and the second author [Kn1, Kn3], who also classifies the exceptional curves on Del Pezzo surfaces.

As for recent work on Green's conjecture we refer to the recent brilliant work of Voisin [Vo1, Vo2], who - most interestingly - uses curves on $K3$ surfaces.

1.3 How the book is organised

Chapter 2. We recall the definition and some basic facts about rational normal scrolls, and how to obtain such scrolls from surfaces with pencils on them.

Most of this stems from [Sc]. At the end we give some special results when the surface is $K3$.

Chapter 3. The Clifford index of a curve is defined and the result of Green and Lazarsfeld for curves on $K3$ surfaces is given. We define the Clifford index of a base point free line bundle L with $L^2 = 2g - 2$ (or the polarized surface (S, L)) to be the Clifford index of all the smooth curves in $|L|$. The divisor class D on S computing the Clifford index c of L, when this is less than $\lfloor \frac{g-1}{2} \rfloor$, is studied, and we show that we can always find one such satisfying $0 \leq D^2 \leq c + 2$ and such that $|D|$ is base point free and the general member of $|D|$ is a smooth curve. Such a divisor (class) will be called a *free Clifford divisor* for L (Definition 3.6). (The definition only depends on the class of D.)

The images of the members of $|D|$ by φ_L span sublinear spaces inside $\mathbf{P}^g$. Each subpencil $\{D_\lambda\}$ within the complete linear system $|D|$ then gives rise to a pencil of sublinear spaces. For each fixed pencil the union of these spaces will be a rational normal scroll $\mathcal{T}$.

Chapter 4. The main result from [Kn2], the above mentioned *Existence Theorem*, and its proof are recalled.

Chapter 5. We study in detail the singular locus of the projective model $\varphi_L(S)$ and the scroll $\mathcal{T}$ in which we choose to view this model as contained. We show (Theorem 5.7) that we can always find a free Clifford divisor D such that the singular locus of $\mathcal{T}$ is "spanned" by the images of the base points of the pencil $\{D_\lambda\}$ and the contractions of smooth rational curves across the members of the pencil. A free Clifford divisor with this extra property will be called a *perfect Clifford divisor* (Definition 5.9). The proofs use results about higher order embeddings of $K3$ surfaces as developed by the second author in [Kn4], which we briefly recall in Section 1.4 below. We also include a study of the projective model if $c = 0$ (the hyperelliptic case), which is Saint-Donat's classical result [SD]. Some proofs are postponed until the next chapter.

Chapter 6. Here some of the longer proofs of the results in the previous chapter are given.

Chapter 7. We study and find (up to certain invariants) a resolution of $\varphi_L(S)$ inside its scroll $\mathcal{T}$ when $\mathcal{T}$ is smooth. In this case a general hyperplane section of $\mathcal{T}$ is a scroll formed in a similar way from a pencil computing the gonality on a canonical curve C of genus g (the gonality is $c+2$). Such scrolls were studied in [Sc], and our results (Lemma 7.1 and Proposition 7.2) for $K3$ surfaces in smooth scrolls are quite parallel to those of [Sc].

Chapter 8. We treat the case when the scroll $\mathcal{T}$ is singular. The approach is to study the blow up $f : \tilde{S} \to S$ at the D^2 base points of the pencil $\{D_\lambda\}$ and the projective model $S'' := \varphi_H(\tilde{S})$ of $\tilde{S}$ by the base point free line bundle $H := f^*L + f^*D - E$, where E is the exceptional divisor. The pencil $|f^*D - E|$ defines a smooth rational normal scroll $\mathcal{T}_0$ that contains S'' and is a desingularization of $\mathcal{T}$.

We use Koszul cohomology and techniques inspired by Green and Lazarsfeld to compute some Betti-numbers of the $\varphi_L(D_\lambda)$ and we obtain that they all have the same Betti-numbers for low values of D^2 and this is a necessary and sufficient condition for "lifting" the resolutions of the fibers to one of the surface S'' in $\mathcal{T}_0$. We prove that S'' is normal, and use this to give more details about the resolution. We give conditions under which we can push down the resolution to one of $\varphi_L(S)$ in $\mathcal{T}$. Here we use results from [Sc]. We end the section by investigating some examples for low genera.

Chapter 9. We consider in more detail the projective models in smooth scrolls for $c = 1$, 2 and 3 ($< \lfloor \frac{g-1}{2} \rfloor$). The description is particularly nice for $c = 1$ and 2, since the projective models are complete intersections in their corresponding scrolls.

We study the sets of projective models in $(c+2)$-dimensional scrolls of given types. Since the scroll type is dependent on which rational curves that exist on S, and therefore on the Picard lattice, it is natural that the dimension of the set of models in question in a scroll as described is dependent on the scroll type. We study this interplay, and obtain a fairly clear picture for $c = 1$ and 2. Most of the information presented can also be obtained from combining material in [Re2], [Ste], and [Br]. For $c = 3$ we study a Pfaffian map of the resolution of $\varphi_L(S)$ in the scroll. In Remark 9.19 we predict the dimension of the set of projective $K3$ models inside a fixed smooth scroll of a given type, for arbitrary $c < \lfloor \frac{g-1}{2} \rfloor$. We state the special case $c = 3$ as Conjecture 9.15.

Chapter 10. We give the definition of BN general polarized $K3$ surfaces introduced by Mukai in [Mu2]: A polarized $K3$ surface (S, L) is said to be *Brill-Noether (BN) general* if for all non-trivial decompositions $L \sim M + N$ one has $h^0(M)h^0(N) < h^0(L)$. (One easily sees that this is for instance satisfied if any smooth curve $C \in |L|$ is Brill-Noether general, i.e. carries no line bundle $\mathcal{A}$ for which $\rho(A) := g - h^0(\mathcal{A})h^1(\mathcal{A}) < 0$.) In [Mu1] it is shown that all such projective models of BN general surfaces of genus $g \leq 10$ and $g = 12$ are complete intersections in certain homogeneous spaces, and that being BN general is also a necessary condition to have such a model (see Theorem 10.3 below).

We study the projective models for $g \leq 10$ that are Clifford general but not BN general. By the concrete description in [Mu2] of such surfaces it follows that their projective models are also contained in scrolls. We analyse them in a similar manner.

Chapter 11. We conclude by giving a complete list and descripton of *all* birational projective models of $K3$ surfaces for $g \leq 10$ (including both the general ones in the sense of Mukai and the remaining ones, that we give a detailed classification of here).

Chapter 12. Some related issues and applications of the ideas developed in this book are discussed, like rational curves in families of Calabi-Yau threefolds and scrolls containing Enriques surfaces.

1.4 Notation and conventions

We use standard notation from algebraic geometry, as in [Hrts].

The ground field is the field of complex numbers. All surfaces are reduced and irreducible *algebraic surfaces.*

By a $K3$ surface is meant a smooth surface S with trivial canonical bundle and such that $H^1(\mathcal{O}_S) = 0$. In particular $h^2(\mathcal{O}_S) = 1$ and $\chi(\mathcal{O}_S) = 2$.

By a *curve* is always meant a *reduced* and *irreducible curve* (possibly singular). The *adjunction formula* for a curve C on a surface S reads $\mathcal{O}_C(C + K_S) \simeq \omega_C$, where ω_C is the dualising sheaf of C, which is just the canonical bundle when C is smooth. In particular, the arithmetic genus p_a of C is given by $C.(C + K_S) = 2p_a - 2$.

On a smooth surface we use line bundles and divisors, as well as the multiplicative and additive notation, with little or no distinction. We denote by Pic S the *Picard group* of S, i.e. the group of linear equivalence classes of line bundles on S. The *Hodge index theorem* yields that if $H \in$ Pic S with $H^2 > 0$, then $D^2H^2 \leq (D.H)^2$ for any $D \in$ Pic S, with equality if and only if $(D.H)H \equiv H^2D$.

Linear equivalence of divisors is denoted by $\sim$, and numerical equivalence by $\equiv$. Note that on a $K3$ surface S linear and numerical equivalence is the same, so that Pic S is torsion free. The usual intersection product of line bundles (or divisors) on surfaces therefore makes the Picard group of a $K3$ surface into a lattice, the *Picard lattice* of S, which we also denote by Pic S.

For two divisors or line bundles M and N on a surface, we use the notation $M \geq N$ to mean $h^0(M-N) > 0$ and $M > N$, if in addition $M-N$ is nontrivial.

If L is any line bundle on a smooth surface, L is said to be *numerically effective*, or simply *nef*, if $L.C \geq 0$ for all curves C on S. In this case L is said to be *big* if $L^2 > 0$.

If $\mathcal{F}$ is any coherent sheaf on a variety V, we shall denote by $h^i(\mathcal{F})$ the complex dimension of $H^i(V, \mathcal{F})$, and by $\chi(\mathcal{F})$ the *Euler characteristic* $\sum(-1)^i h^i(\mathcal{F})$. In particular, if D is any divisor on a normal surface S, the *Riemann-Roch formula* for D is $\chi(\mathcal{O}_S(D)) = \frac{1}{2}D.(D-K_S)+\chi(\mathcal{O}_S)$. Moreover, if D is effective and nonzero and $\mathcal{L}$ is any line bundle on D, the Riemann-Roch formula for $\mathcal{L}$ on D is $\chi(\mathcal{L}) = \deg \mathcal{L} + 1 - p_a(D) = \deg \mathcal{L} - \frac{1}{2}D.(D + K_S)$.

We will make use of the following results of Saint-Donat on line bundles on $K3$ surfaces. The first result will be used repeatedly, without further mention.

Proposition 1.8. [SD, Cor. 3.2] *A complete linear system on a $K3$ surface has no base points outside of its fixed components.*

Proposition 1.9. [SD, Prop. 2.6] *Let L be a line bundle on a $K3$ surface S such that $|L| \neq \emptyset$ and such that $|L|$ has no fixed components. Then either*

(i) $L^2 > 0$ and the general member of $|L|$ is a smooth curve of genus $L^2/2+1$. In this case $h^1(L) = 0$, or
(ii) $L^2 = 0$, then $L \simeq \mathcal{O}_S(kE)$, where k is an integer ≥ 1 and E is a smooth curve of arithmetic genus 1. In this case $h^1(L) = k-1$ and every member of $|L|$ can be written as a sum $E_1+\cdots+E_k$, where $E_i \in |E|$ for $i = 1, \ldots, k$.