Studies in Algebra and Number Theory

ADVANCES IN MATHEMATICS
SUPPLEMENTARY STUDIES, VOLUME 6

EDITED BY
Gian-Carlo Rota

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Preface

The supplementary volumes of the journal *Advances in Mathematics* are issued from time to time to facilitate publication of papers already accepted for publication in the journal. The volumes will deal in general—but not always—with papers on related subjects, such as algebra, topology, foundations, etc., and are available individually and independently of the journal.

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Selberg's Trace Formula for Nonuniform Lattices: The R-Rank One Case†

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1. INTRODUCTION

Let G be a noncompact connected simple Lie group of split-rank 1; let Γ be a discrete subgroup of G such that the volume of G/Γ is finite but such that G/Γ is not compact. For example, the pair (G, Γ) where $G = \mathbf{SL}(2, \mathbf{R})$, $\Gamma = \mathbf{SL}(2, \mathbf{Z})$ satisfies these hypotheses. Call $L_{G/\Gamma}$ the left regular representation of G on the Hilbert space $L^2(G/\Gamma)$—then a central problem in the theory of automorphic forms relative to the pair (G, Γ) is the decomposition of $L_{G/\Gamma}$ into irreducible unitary representations. As a first step one proves, using the theory of Eisenstein series, that $L^2(G/\Gamma)$ admits an orthogonal decomposition

$$L^2(G/\Gamma) = L_{\mathrm{d}}^2(G/\Gamma) \oplus L_{\mathrm{c}}^2(G/\Gamma),$$

$L_{\mathrm{d}}^2(G/\Gamma)$ (respectively $L_{\mathrm{c}}^2(G/\Gamma)$) being an $L_{G/\Gamma}$-invariant subspace of $L^2(G/\Gamma)$ in which $L_{G/\Gamma}$ decomposes discretely (respectively continuously). Call $L_{G/\Gamma}^{\mathrm{d}}$ (respectively $L_{G/\Gamma}^{\mathrm{c}}$) the restriction of $L_{G/\Gamma}$ to $L_{\mathrm{d}}^2(G/\Gamma)$ (respectively $L_{\mathrm{c}}^2(G/\Gamma)$). It turns out that $L_{G/\Gamma}^{\mathrm{c}}$ can be written as a direct integral over the "principal

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series" representations of G, so one's understanding of $L^{\mathrm{c}}_{G/\Gamma}$ is essentially complete. As for $L^{\mathrm{d}}_{G/\Gamma}$, if $\hat{G}$ is the set of unitary equivalence classes of irreducible unitary representations of G and if m_U is the multiplicity with which a given U in $\hat{G}$ occurs in $L^{\mathrm{d}}_{G/\Gamma}$ (necessarily finite), then

$$L^{\mathrm{d}}_{G/\Gamma} = \sum_{U \in \hat{G}} \oplus\, m_U U.$$

To describe $L^{\mathrm{d}}_{G/\Gamma}$, therefore, one must determine those U for which $m_U > 0$ together with an explicit formula for m_U. Apart from a few numerical examples, nothing is known about this important problem. One method of attack is to develop a "trace formula" of "Selberg type"; it is to this question that the present article is addressed. The basic idea behind what is going on here is not difficult to describe; on the other hand, the actual execution of the method and justification of the details is rather lengthy. Suppose that α is a smooth integrable function on G—then

$$L_{G/\Gamma}(\alpha) = \int_G \alpha(x) L_{G/\Gamma}(x)\, d_G(x)$$

is an integral operator on $L^2(G/\Gamma)$ which, however, need not be of the trace class. Write $L^{\mathrm{d}}_{G/\Gamma}(\alpha)$ (respectively $L^{\mathrm{c}}_{G/\Gamma}(\alpha)$) for the restriction of $L_{G/\Gamma}(\alpha)$ to $L^2_{\mathrm{d}}(G/\Gamma)$ (respectively $L^2_{\mathrm{c}}(G/\Gamma)$)—then

$$L_{G/\Gamma}(\alpha) = L^{\mathrm{d}}_{G/\Gamma}(\alpha) + L^{\mathrm{c}}_{G/\Gamma}(\alpha).$$

If α is sufficiently regular (say in an appropriate Schwartz space in the sense of Harish-Chandra), then both $L^{\mathrm{d}}_{G/\Gamma}(\alpha)$ and $L^{\mathrm{c}}_{G/\Gamma}(\alpha)$ are integral operators; moreover, it can be shown that $L^{\mathrm{d}}_{G/\Gamma}(\alpha)$ is of the trace class, the trace being computable by integrating its kernel over the diagonal. We shall refer to this result as the first stage of the Selberg trace formula. The next step in the analysis is the computation of the integral giving the trace of $L^{\mathrm{d}}_{G/\Gamma}(\alpha)$. A key role here is played by the Poisson summation formula. The net result is that the trace of $L^{\mathrm{d}}_{G/\Gamma}(\alpha)$ can be expressed in terms of certain distributions on G, e.g., orbital integrals (or perturbations thereof). This is the Selberg trace formula in its second stage. The third and final stage of the Selberg trace formula consists in the explicit determination of the Fourier transforms, in the sense of Harish-Chandra, of the aforementioned distributions. This is the most difficult step in the analysis. For general G, we shall only be able to give a complete answer in the special case when α is bi-invariant under a maximal compact subgroup; this will suffice, though, for certain important applications which will be considered elsewhere. It should be stressed that the only obstacle to having a satisfactory theory in general is the computation of the Fourier transform of a single, albeit complicated, noncentral distribution. Once this has been done, a number of important consequences will follow, e.g., explicit formulas for the multiplicities of the integrable discrete series in $L^2_d(G/\Gamma)$.

The thrust of the present paper, then, is to give a complete and detailed proof of the Selberg trace formula for nonuniform lattices Γ in a simple split-rank 1 group G, at least through the second stage. The investigation depends heavily on the Garland–Raghunathan reduction theory and the theory of Eisenstein series, both of which are reviewed in Section 2. In Section 3 we discuss the spectral decomposition of $L^2(G/\Gamma)$, establishing in particular the orthogonal decomposition

$$L^2(G/\Gamma) = L^2_d(G/\Gamma) \oplus L^2_c(G/\Gamma)$$

mentioned above. Sections 4–6 are devoted to the proof of the Selberg trace formula in its second stage (subject to a certain technical restriction on Γ). The analysis in Section 6 is carried out under the assumption that α has compact support. For the applications, it is necessary to know that the Selberg trace formula is valid for suitable classes of noncompactly supported functions, e.g., the K-finite matrix coefficients of the integrable discrete series. Such an extension is made in Section 8. Section 7, which is of a preliminary nature, serves to establish the convergence of certain Epstein-like zeta functions. In Section 9, the Selberg trace formula in its third stage is developed for "class one" functions. We terminate in Section 10 with a list of open problems and indicate a number of avenues for further research.

At this point it is perhaps appropriate to make some remarks of a historical nature. The whole subject originated with Selberg's [27a] famous paper (although Delsarte had apparently anticipated some of the ideas years before this). Selberg only made explicit statements about $\mathbf{SO}(2)\backslash\mathbf{SL}(2,\mathbf{R})/\Gamma$ and did not give any proofs there. Selberg did give, however, complete proofs for this case in an unpublished manuscript which has had a fairly wide circulation. Independently, proofs have been provided in this special situation by a number of people, including (at least) Faddev, Kalinin and Venkov, Kubota, Lax and Phillips, Langlands, and the author. No progress of any real significance was made for some fifteen years until the work of Langlands appeared (cf. Jacquet and Langlands [16]). Langlands deals with $\mathbf{GL}(2)$ in the adele picture and gives a comprehensive outline of how the trace formula should go in that setting; complete details were later supplied by Duflo and Labesse [6]. Langlands' methods differ somewhat in detail from those of Selberg (although not, of course, in spirit) and are more susceptible to generalization. They were in fact developed by Arthur [1a] in the adele picture for semisimple algebraic groups of rank 1 over a number field. Much of our treatment is directly motivated by the work of Arthur and Langlands. Finally, we should mention that Venkov [32] has recently studied the case $\mathbf{SO}(n)\backslash\mathbf{SO}(n,1)/\Gamma$.

2. Eisenstein Series

Let G be a noncompact connected semisimple Lie group with finite center; let K be a maximal compact subgroup of G. We shall assume that $\operatorname{rank}(G/K) = 1$. In addition it will be supposed that G is simple and is embedded in the simply connected complex analytic group corresponding to the complexification of the Lie algebra $\mathfrak{g}$ of G.

Let Γ be a discrete subgroup of G such that the volume of G/Γ is finite but such that G/Γ is not compact. Under these circumstances, the reduction theory of Garland and Raghunathan [11, pp. 304–306] is applicable and may be described as follows. Relative to some Iwasawa decomposition $G = K \cdot A \cdot N$ of G, there is a parabolic subgroup P of G with Langlands decomposition $P = M \cdot A \cdot N$ (M the centralizer of A in K) having certain properties which we shall now enumerate. Let λ be the unique simple root of the pair (G, A) implicit in the choice of N; let $\xi_\lambda : A \to \mathbf{R}^+$ be the associated quasi-character of A. Given $t > 0$, put

$$A[t] = \{a \in A : \xi_\lambda(a) \leqslant t\}, \qquad A(t) = \{a \in A : \xi_\lambda(a) < t\}.$$

For any compact neighborhood ω of 1 in N, the set $\mathfrak{S}_{t,\omega} = K \cdot A[t] \cdot \omega$ is called a *Siegel domain* in G (relative to P) while the set $\mathfrak{C}_t = K \cdot A(t) \cdot N$ is called a *cylindrical domain* in G (relative to P). Let r be the number of Γ-inequivalent cusps—then one can choose elements $k_1 = 1, \ldots, k_r$ in K such that the conjugates $P_i = {}^{k_i}P$ form a complete set of representatives for the Γ-cuspidal parabolic subgroups of $G \bmod \Gamma$. Each P_i admits a Langlands decomposition $P_i = M_i \cdot A_i \cdot N_i$ where $M_i = {}^{k_i}M$, $A_i = {}^{k_i}A$, $N_i = {}^{k_i}N$ ($i = 1, \ldots, r$). Let $\kappa_i = k_i^{-1}$, $\mathfrak{s} = \{\kappa_i\}$—then one can find a Siegel domain $\mathfrak{S}_{t_0,\omega_0}$ such that the set $\mathfrak{S} = \mathfrak{S}_{t_0,\omega_0} \cdot \mathfrak{s}$ has the following properties:

(i) $\mathfrak{S} \cdot \Gamma = G$;
(ii) $\{\gamma \in \Gamma : \mathfrak{S}\gamma \cap \mathfrak{S} \neq \varnothing\}$ is finite.

We remark that once t_0 and ω_0 have been shown to exist, they can then be replaced by any $t > t_0$, $\omega \supset \omega_0$ without affecting either statement (i) or statement (ii). For $1 \leqslant i \leqslant r$,

$$N_i/N_i \cap \Gamma$$

is compact. Because $\mathfrak{s}$ is finite, it can be assumed that ω_0 is chosen in such a way that $k_i\omega_0 k_i^{-1} = \kappa_i^{-1}\omega_0\kappa_i$ contains a fundamental domain for the group $N_i \cap \Gamma$ acting to the right on N_i. Using this hypothesis on ω_0, one can then produce a ${}_0t < t_0$ such that:

(iii) $K \cdot A[{}_0t] \cdot \omega_0 \cdot \kappa_i \cap K \cdot A[{}_0t] \cdot \omega_0 \cdot \kappa_j\gamma = \varnothing$ for $\kappa_i, \kappa_j \in \mathfrak{s}$ ($i \neq j$) and for $\gamma \in \Gamma$;

(iv) $K \cdot A[_0t] \cdot \omega_0 \cdot \kappa_i \cap K \cdot A[_0t] \cdot \omega_0 \cdot \kappa_i\gamma \neq \varnothing$ for $\kappa_i \in \mathfrak{s}$ and for $\gamma \in \Gamma$, only if $\gamma \in M_i \cdot N_i$.

Speaking roughly, these two properties say that it is possible to separate the cusps of Γ. A simple argument (cf. Raghunathan [23a, p. 289]) then gives that the sets

$$K \cdot A[_0t] \cdot N\kappa_i \qquad (1 \leq i \leq r)$$

are mutually disjoint and that, moreover, for any i

$$K \cdot A[_0t] \cdot N\kappa_i \cap K \cdot A[_0t] \cdot N\kappa_i\gamma \neq \varnothing$$

if and only if $\gamma \in M_i \cdot N_i$. Consequently (cf. Raghunathan [23a, p. 290]) there exists a compact subset Ω_{0t} of G such that the complement of the Γ-saturation $\Omega_{0t} \cdot \Gamma$ of Ω_{0t} in G decomposes into a finite number of mutually disjoint Γ-saturated open sets $\mathfrak{D}_1, \ldots, \mathfrak{D}_r$ where each $\mathfrak{D}_i$ contains $\mathfrak{C}_{0t}\kappa_i$ as an open and closed subset and is in fact the Γ-saturation of $\mathfrak{C}_{0t}\kappa_i$. In other words

$$G = \Omega_{0t} \cdot \Gamma \cup \bigcup_{i=1}^{r} \mathfrak{C}_{0t}\kappa_i \cdot \Gamma \qquad \text{(disjoint union)},$$

so

$$G/\Gamma = \pi(\Omega_{0t}) \cup \bigcup_{i=1}^{r} \pi(\mathfrak{C}_{0t}\kappa_i) \qquad \text{(disjoint union)},$$

where $\pi: G \to G/\Gamma$ is the natural projection. As a final point in this circle of ideas, we mention that the set of conjugacy classes of maximal unipotent subgroups of Γ is finite and, in fact, that every maximal unipotent subgroup of Γ is conjugate to a unique $N_i \cap \Gamma$; for details, see Raghunathan [23b, p. 202].

We turn now to the theory of Eisenstein series on G/Γ. Complete proofs can be found in Harish-Chandra [13a] or Langlands [19a]. Harish-Chandra [13a] explicitly treats only the case when Γ is arithmetic; this is done because Borel's reduction theory is then applicable. But, using the Garland–Raghunathan reduction theory, one can carry over the theory virtually word for word to the general case. Alternatively, it is easy to verify that the Garland–Raghunathan reduction theory implies that the axioms assumed by Langlands [19a] are in force in the present case so that one can quote Langlands [19a] directly.

Keeping to the above notations, identify M with $M \cdot N/N$. It is known that $\Gamma \cap P \subset M \cdot N$ (cf. Garland and Raghunathan [11, p. 295]). Put $\Gamma_M = \Gamma \cap M \cdot N/\Gamma \cap N$—then Γ_M is a discrete subgroup of M, hence is finite, M being compact. Let M^* be the normalizer of A in K—then $W(A) = M^*/M$ is the Weyl group of the pair (G, A). Since $\operatorname{rank}(G/K) = 1$, $W(A)$ is of order 2. Let $\hat{M}$ be the set of unitary equivalence classes of irreducible unitary representations of M. For each $\sigma \in \hat{M}$, let ξ_σ denote the character of σ, $d(\sigma)$ the degree of σ, and $\chi_\sigma = d(\sigma)\xi_\sigma$. The group $W(A)$ operates to the left on $\hat{M}$ in the obvious way. Write

$$L^2(M/\Gamma_M) = \sum_{\sigma \in \hat{M}} \oplus\, n(\sigma, \Gamma_M)\sigma,$$

the $n(\sigma, \Gamma_M)$ being certain nonnegative integers. Given an orbit ϑ in $W(A)\backslash\hat{M}$, pick $\sigma \in \vartheta$ and set

$$S_\vartheta = n(\sigma, \Gamma_M)\sigma + n(w\sigma, \Gamma_M)w\sigma,$$

w the nontrivial element in $W(A)$. Then

$$L^2(M/\Gamma_M) = \sum_{\vartheta \in W(A)\backslash\hat{M}} \oplus\, S_\vartheta.$$

Let $\hat{K}$ be the set of unitary equivalence classes of irreducible unitary representations of K. For each $\delta \in \hat{K}$, let ξ_δ denote the character of δ, $d(\delta)$ the degree of δ, and $\chi_\delta = d(\delta)\xi_\delta$. Fix a class $\delta \in \hat{K}$. Denote by $L^2(K;\delta)$ the subspace of $L^2(K)$ consisting of those functions which transform under the left regular representation according to δ. Given $\vartheta \in W(A)\backslash\hat{M}$ such that S_ϑ is nontrivial, let $\mathscr{E}(\vartheta, \delta)$ be the set of all continuous functions $\Phi: G \to \mathbf{C}$ such that:

(i) Φ is right invariant under $(\Gamma \cap P) \cdot A \cdot N$;
(ii) for every $x \in G$, the function

$$m \mapsto \Phi(xm)$$

belongs to S_ϑ;
(iii) for every $x \in G$, the function

$$k \mapsto \Phi(kx)$$

belongs to $L^2(K;\delta)$.

It is known that $\mathscr{E}(\vartheta, \delta)$ is a finite-dimensional Hilbert space of analytic functions with inner product

$$(\Phi, \Psi) = \int_K \int_{M/\Gamma_M} \Phi(km)\overline{\Psi(km)}\, d_K(k)\, d_M(m).$$

This is proved formally in Langlands [19a, p. 50] and is actually a consequence of some observations made in the next section.

Let $\mathfrak{a}$ be the Lie algebra of A—then we shall agree to equip the dual of $\mathfrak{a}$ with the usual Euclidean structure derived from the Killing form. Since $\lambda/|\lambda|$ is a unit vector for this structure, a complex number s becomes a linear function on $\mathfrak{a}$ through the identification $s \leftrightarrow s(\lambda/|\lambda|)$. In particular ρ, the sum of the positive roots of the pair (G, A) (counted with multiplicity) divided by 2, is identified with its length $|\rho|$. For any $x \in G$, we denote by $H(x)$ that element of $\mathfrak{a}$ such that $x \in K \exp(H(x)) \cdot N$. This said, let $\Phi \in \mathscr{E}(\vartheta, \delta)$—then, attached to Φ is the *Eisenstein series*

$$E(P:\Phi:s:x) = \sum_{\gamma \in \Gamma/\Gamma \cap P} e^{(s-|\rho|)(H(x\gamma))}\Phi(x\gamma).$$

$E(P:\Phi:s:x)$ is a C^∞ function on $\{s: \mathrm{Re}(s) < -|\rho|\} \times G$ which is holomorphic in s and right invariant under Γ. For every $x \in G$, the function

$$k \mapsto E(P:\Phi:s:kx) \qquad (\mathrm{Re}(s) < -|\rho|)$$

belongs to $L^2(K;\delta)$. If $\mathfrak{Z}$ is the center of the universal enveloping algebra $\mathfrak{G}$ of $\mathfrak{g}_c$ ($\mathfrak{g}_c$ the complexification of the Lie algebra $\mathfrak{g}$ of G), then $E(P:\Phi:s:x)$ is $\mathfrak{Z}$-finite. All these assertions are detailed in Harish-Chandra [13a, pp. 26–31].

Modulo obvious notational changes, the definitions and results indicated above carry over to each of the P_i $(i = 1, \ldots, r)$. Fix i and j—then $P_i = M_i \cdot A_i \cdot N_i$, $P_j = M_j \cdot A_j \cdot N_j$ and the orbit spaces

$$W(A_i)\backslash\hat{M}_i, \qquad W(A_j)\backslash\hat{M}_j$$

are in a canonical one-to-one correspondence. Corresponding orbits are said to be *associate*. Introduce the set $W(A_i, A_j)$ of all bijections $w: A_i \to A_j$ such that $wa_i = xa_ix^{-1}$ $(a_i \in A_i)$ for some $x \in G$. Now fix $\delta \in \hat{K}$ and associate orbits $\vartheta_i \in W(A_i)\backslash\hat{M}_i$, $\vartheta_j \in W(A_j)\backslash\hat{M}_j$. Let $\Phi_i \in \mathscr{E}(\vartheta_i, \delta)$—then the integral

$$\int_{N_j/N_j \cap \Gamma} E(P_i:\Phi_i:s:xn_j)\, d_{N_j}(n_j),$$

known as the *constant term* of the Eisenstein series $E(P_i:\Phi_i:s:x)$ along P_j and denoted by $E_{P_j}(P_i:\Phi_i:s:x)$, is computable and in fact

$$E_{P_j}(P_i:\Phi_i:s:x) = \sum_{w \in W(A_i,A_j)} e^{(ws-|\rho|)(H_j(x))} \cdot (c_{P_j|P_i}(w:s)\Phi_i)(x).$$

Here

$$c_{P_j|P_i}(w:s): \mathscr{E}(\vartheta_i, \delta) \to \mathscr{E}(\vartheta_j, \delta)$$

is a certain linear transformation which, as a function of s, is defined and holomorphic in the region $\mathrm{Re}(s) < -|\rho|$ (cf. Harish-Chandra [13a, p. 44]). Let $\vartheta = \vartheta_1, \vartheta_2, \ldots, \vartheta_r$ be a complete collection of associate orbits. Put

$$\mathscr{E}(\vartheta, \delta) = \sum_{i=1}^{r} \oplus\, \mathscr{E}(\vartheta_i, \delta),$$

where the column vector

$$\mathbf{\Phi} = \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_r \end{pmatrix} \in \mathscr{E}(\vartheta, \delta)$$

has norm $\|\mathbf{\Phi}\|^2 = \sum_{i=1}^r \|\Phi_i\|^2$. For any complex number s such that $\mathrm{Re}(s) < -|\rho|$ we want to define a linear transformation

$$\mathbf{c}_{\vartheta,\delta}(s): \mathscr{E}(\vartheta, \delta) \to \mathscr{E}(\vartheta, \delta).$$

This is done as follows. Let

$$\mathbf{\Phi} = \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_r \end{pmatrix} \in \mathscr{E}(\vartheta, \delta).$$

Then it is enough to define $\mathbf{c}_{\vartheta,\delta}(s)\Phi_i$ $(i = 1, \ldots, r)$, which in turn is completely prescribed when $(\mathbf{c}_{\vartheta,\delta}(s)\Phi_i)_j$ is defined $(j = 1, \ldots, r)$. Put

$$(\mathbf{c}_{\vartheta,\delta}(s)\Phi_i)_j = c_{P_j|P_i}(k_j w k_i^{-1} : k_i s)\Phi_i \in \mathscr{E}(\vartheta_j, \delta),$$

w the nontrivial element in $W(A)$. Interchanging i and j, we thus have

$$\begin{aligned}(\mathbf{c}_{\vartheta,\delta}(s)\mathbf{\Phi})_i &= \left(\sum_j \mathbf{c}_{\vartheta,\delta}(s)\Phi_j\right)_i \\ &= \sum_j c_{P_i|P_j}(k_i w k_j^{-1} : k_j s)\Phi_j,\end{aligned}$$

which can be interpreted as saying that the image of $\mathbf{\Phi}$ under $\mathbf{c}_{\vartheta,\delta}(s)$ is simply obtained by formal matrix multiplication. Fundamental to the theory is the fact that $\mathbf{c}_{\vartheta,\delta}$ can be meromorphically continued to the whole s-plane. Its poles in the half-plane $\mathrm{Re}(s) < 0$ lie in the set $\{s \in \mathbf{R} : -|\rho| \leqslant s < 0\}$; there are but finitely many of them there and they are all simple. Along the imaginary axis, $\mathbf{c}_{\vartheta,\delta}$ is holomorphic. Another basic point is that $\mathbf{c}_{\vartheta,\delta}$ satisfies a functional equation, viz.

$$\mathbf{c}_{\vartheta,\delta}(s)\mathbf{c}_{\vartheta,\delta}(-s) = I,$$

I the identity operator. Because the adjoint $\mathbf{c}_{\vartheta,\delta}(s)^*$ of $\mathbf{c}_{\vartheta,\delta}(s)$ is $\mathbf{c}_{\vartheta,\delta}(\bar{s})$, it follows that if s is pure imaginary, then

$$\mathbf{c}_{\vartheta,\delta}(s)\mathbf{c}_{\vartheta,\delta}(-s) = \mathbf{c}_{\vartheta,\delta}(s)\mathbf{c}_{\vartheta,\delta}(\bar{s}) = \mathbf{c}_{\vartheta,\delta}(s)\mathbf{c}_{\vartheta,\delta}(s)^* = I,$$

so $\mathbf{c}_{\vartheta,\delta}$ is unitary along the imaginary axis. Given

$$\mathbf{\Phi} = \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_r \end{pmatrix} \in \mathscr{E}(\vartheta, \delta),$$

put

$$\mathbf{E}(\mathbf{\Phi} : s : x) = \sum_{i=1}^r E(P_i : \Phi_i : s : x).$$

One can prove that $\mathbf{E}(\mathbf{\Phi}:s:x)$, as a function of s, can be meromorphically continued to the whole s-plane; moreover, its poles are poles of $\mathbf{c}_{\vartheta,\delta}$. Finally,

$$\mathbf{E}(\mathbf{\Phi}:s:x) = \mathbf{E}(\mathbf{c}_{\vartheta,\delta}(s)\mathbf{\Phi}:-s:x).$$

The proofs of the above results are set down in Harish-Chandra [13a, pp. 89–105].

Eisenstein series, while right invariant under Γ, do not lie in $L^2(G/\Gamma)$. Their significance in the spectral decomposition of $L^2(G/\Gamma)$ will be explained in the next section. To produce functions in $L^2(G/\Gamma)$, it is convenient to introduce the notion of theta series. This will now be done.

Fix an orbit $\vartheta \in W(A)\backslash\hat{M}$ such that S_ϑ is nontrivial. Fix a class $\delta \in \hat{K}$. Let $\phi: G \to \mathbf{C}$ be a differentiable function such that:

(i) ϕ is right invariant under $(\Gamma \cap P) \cdot N$;

(ii) for every $x \in G$, the function

$$m \mapsto \phi(xm)$$

belongs to S_ϑ;

(iii) for every $x \in G$, the function

$$k \mapsto \phi(kx)$$

belongs to $L^2(K;\delta)$.

Then one can associate with ϕ a differentiable function

$$\check{\phi}: A \to \mathscr{E}(\vartheta,\delta),$$

that is, a differentiable function

$$\check{\phi}: A \times G \to \mathbf{C}$$

with the property that for each $a \in A$, the function $x \mapsto \check{\phi}(a:x)$ belongs to $\mathscr{E}(\vartheta,\delta)$. Explicitly, if k_x is the K-component of x in the Iwasawa decomposition $G = K \cdot A \cdot N$, then $\check{\phi}(a:x) = \phi(k_x a)$. The set of all ϕ for which $\check{\phi}$, as a function from A to $\mathscr{E}(\vartheta,\delta)$, is of compact support is denoted by $\mathscr{V}(\vartheta,\delta)$. It can be shown without difficulty that the correspondence $\phi \leftrightarrow \check{\phi}$ serves to identify $\mathscr{V}(\vartheta,\delta)$ with $C_c^\infty(A) \otimes \mathscr{E}(\vartheta,\delta)$. Let $\widehat{C_c^\infty(A)}$ be the set of Fourier–Laplace transforms of $C_c^\infty(A)$. Suppose that $\phi \in \mathscr{V}(\vartheta,\delta)$—then there is associated with ϕ in a canonical way an element $\hat{\phi} \in \widehat{C_c^\infty(A)} \otimes \mathscr{E}(\vartheta,\delta)$, called the *Fourier transform* of ϕ, such that

$$\phi(x) = \frac{1}{2\pi}\int_{\mathrm{Re}(s)=s_0} \hat{\phi}(s:x)e^{(s-|\rho|)(H(x))}|ds|,$$