N. KEMMER

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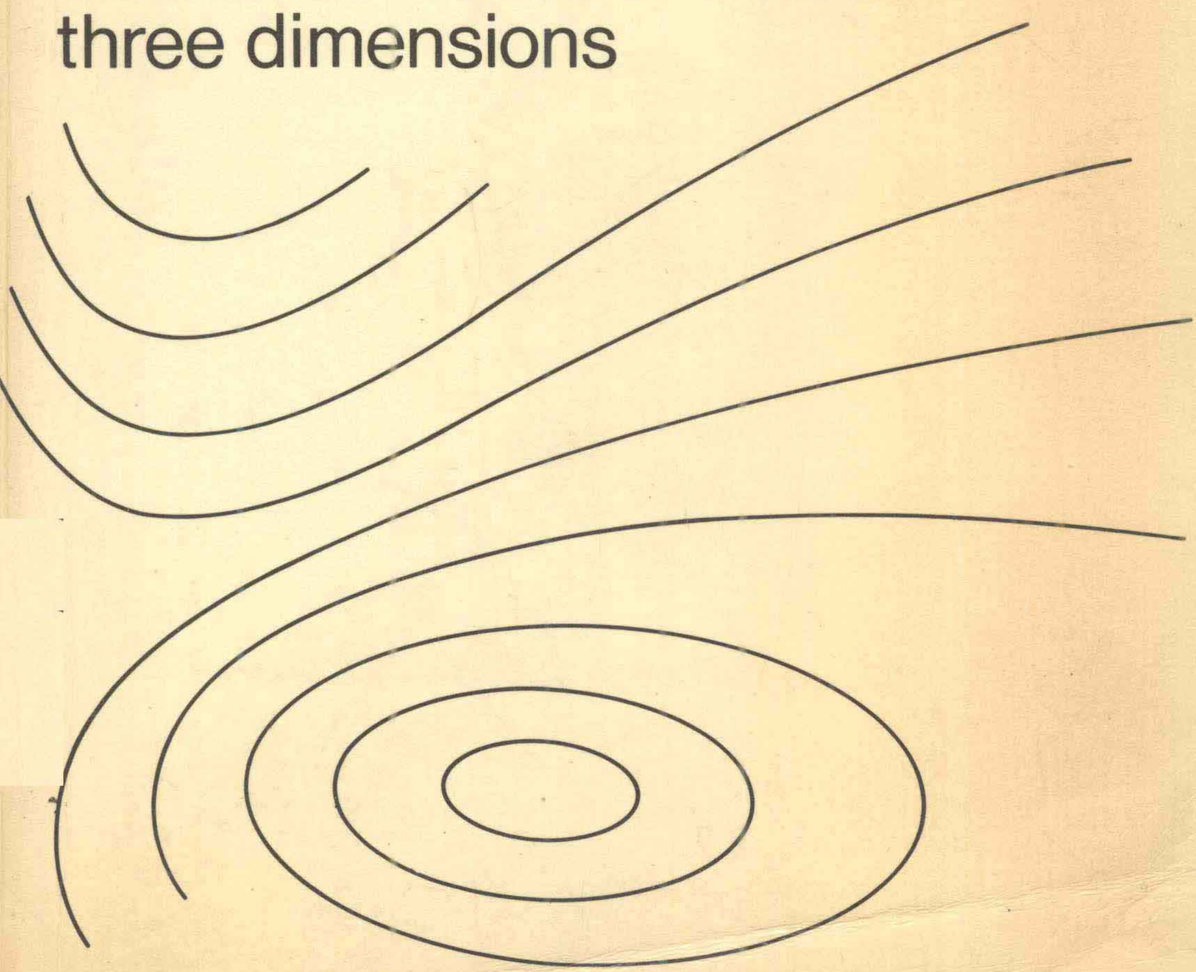
A physicist's guide to the mathematics of fields in three dimensions



Vector analysis

A physicist's guide to the mathematics of fields in three dimensions

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Preface

The justification for adding one more to the many available texts on vector analysis cannot be novelty of content. In this book I have included a few topics that are more frequently encountered as part of the discussion of a specific physical topic – in the main in fluid dynamics or electrodynamics – but this in itself does not justify telling the whole story over again, perhaps least of all in the classical way – in bold-type vectors, excluding suffixes, tensors, n-dimensional space etc. However, after giving a course in the standard shape for many years (to Scottish students in their second year – roughly comparable in level to an English first-year University course) I gradually developed a way of presenting the subject which gave the old tale a new look and seemed to me to make a more coherent whole of it. Once my course had taken this shape the students were left without a suitable text to work from. More and more *ad hoc* handouts became necessary and finally the skeleton of this book emerged.

In preparation for publication I spent more time on devising the sections of exercises than on the main text which was largely in existence from the start. The exercises were indeed devised rather than compiled and I am well aware that they reveal my idiosyncrasies. I see them as a necessary part of learning a language. Their nature changes from the 'plume de ma tante' level to excerpts from the Classics and the Moderns – interspersed with lists of idioms. Predominantly they require a hard grind, which I have tried to enliven to my taste. The student who is uncertain about integrating anything beyond x^n will not be at a particular disadvantage, but one who fears algebraic manipulation will face a challenge – though not without words of encouragement in appropriate places. He will also be shown by example the advantage of keeping his examples strictly orderly and he should certainly become increasingly aware of how much symmetries help one to simplify calculations.

I must remark particularly on exercises A after chapter 1 where I have amused myself by providing – sometimes with considerable effort – vector relations involving integers only. If I had been seeking physical realism at that stage this would have been utterly inappropriate. In defence I would say that in this age of the pocket calculator *any* inducement to doing arithmetic by hand should be welcome.

As the whole concept of my presentation is linked to the visualisation of parameterised curved surfaces, their edge curves and the spaces enclosed by them, a main condition for the conversion of my lectures into a book was adequate illustration. I have been extremely fortunate in getting support from an artist who translated the rough sketches used in my course into superb illustrations. I am very happy to express here my deep gratitude to Mr P. Flook for this beautiful artwork.

I have special pleasure in expressing my sincerest thanks to my secretary, Mrs Rachel Chester. The preparation of this book would have been delayed by many months, if not altogether frustrated, without her expert and devoted assistance. Through all stages of this venture she has produced the typescript of every page of draft and final manuscript – all the text and all the formulae.

My sincere thanks go also to all members of the staff of the Cambridge University Press, who have proved most helpful at all stages of the production of this book.

N.K.

May 1976

Introduction

If one wishes to express in precise and general terms any statement in physics in which positions, directions and motions in space are involved the most appropriate language to use is the language of vectors. In the mechanics of particles and rigid bodies vectors are used extensively and it is assumed in this monograph that the reader has some prior knowledge of vector *algebra*, which is the part of vector theory required in mechanics. Nevertheless chapter 1 provides a summary of vector algebra. There the notation to be used is made explicit and a brief survey of the whole field is given with stress laid on a number of particular results that become especially important later. The reader is also given the opportunity to test his understanding of vector algebra and his facility in applying it to detailed problems: a fairly extensive set of examples (exercises A) follows the chapter, with some comments and answers provided at the end of the book.

A considerably widened theory of vectors becomes necessary when one turns to such parts of physics as fluid dynamics and electromagnetic theory where one deals not just with things at certain particular points in space but with the physical objects as distributed continuously in space. Quantities that are continuous functions of the coordinates of a general point in space are called *fields* and some of the fields of greatest interest in physics are *vector fields*. How such vector fields can be described and interrelated by using the methods of integral and differential calculus is the theme of the book.

After the introductory chapter 1 there are two further chapters in which vector fields are not yet mentioned. These deal with geometrical preliminaries. Within mathematics there is an important subject, differential geometry, in which among many other things the topics of these two chapters are fully covered. Here we need only some selected results and these are summarised in precisely the form wanted for later use. Two non-standard definitions are introduced: the 'quasi-square' and the 'quasi-cube'. A great deal of the later analysis depends on these and it is hoped that readers will not skip this preliminary work. The exercises B are designed to familiarise the student thoroughly with these ideas. The illustrations in the main text should also help to drive home the important ideas.

Chapter 4 opens the discussion of vector fields and deals with ideas which are introduced at the very beginning in other books. The reason why a rather slower approach to them has been chosen is because in this way one can develop the whole of vector analysis proper without introducing any further mathematical machinery on the way. The earlier parts of some other presentations may be simpler, but then some later mathematical points must either be taken on trust or be allowed to interrupt the more central part of the development. Since this text is intended primarily for physicists, *full* mathematical rigour is not aimed at, at any stage; but the assumptions that are necessary to justify the way we proceed later will all have been stated by the end of chapter 3. As far as the mathematics goes the reader will be virtually coasting down hill from then on.

The real core of the book will be found in chapters 5–9; in particular the last three of these chapters form a sub-unit within which the powerful central theorems of vector analysis are established. Then follow the chapters 10 and 11 which are in the nature of addenda, containing material which is not always included as part of the subject. Chapter 12 is then somewhat transitional, leading up to the final three chapters which deal with topics closely related to the central theme but involving other ideas as well. In this part it had to be decided what was worth including and the decisions were strongly influenced by the thought that many lecture courses on physical topics could benefit from saving of time often spent on mathematical preliminaries, which cannot be found so easily in books written for physicists.

The sets of exercises associated with the earlier parts of the book – including the central chapters – will be found to be unconnected with physical applications. They are intended to help the student to acquire a clear picture of the properties of *any* vector fields and not only the special ones that are later of importance. But when it comes to the later groups of exercises, ideas related to particular physical situations are introduced. From these the reader should get a first impression of how the language of vectors helps to make physics the exact science that it is.

At the beginning of each group of exercises there are comments on the contents of the group. The student should bear these in mind and remember that he is not expected to work slavishly through all the exercises. Different people will feel the need for different amounts of explicit work before they feel they understand a technique. The exercises try to cater even for the most dogged characters. But it is hoped that some of the exercises will be attempted by all readers – with or without help from the 'Answers and comments' at the end of the book.

There is a fair amount of interconnection between the different groups of exercises. In a number of cases the student will encounter the same topic or problem treated with increasing sophistication as the subject is developed and what can only be worked out with hard labour at an earlier stage may be seen as a simple consequence of a general theorem later. Also, the relevance of a particular result to physics may emerge gradually.

However, while it is hoped that practising on these examples will benefit the serious student considerably, let him not be under the illusion that solving them will be *directly* useful to him in answering standard British examination questions on vector analysis. Very few indeed of the exercises are of the traditional format. A few are too brief – very many more are too long and heavy. None have been taken from old examination papers. Understanding of how to solve these exercises will surely be helpful; imitation would be dangerous!

Finally a word about related subjects. There are many fields in physics, foremost among them the special and general theories of relativity, in which the limitations of this book – the restriction to space of not more than three dimensions and the exclusion of tensors (and of spinors) – are unacceptable. In more advanced disciplines one must not only widen the range of mathematical objects utilised but also change drastically the form in which these objects are described: tensor analysis not only is but also *looks* quite different from vector analysis. But this does not mean that the methods of this book deserve to be forgotten and replaced by more general ones. There are still many parts of physics in which vector methods remain peculiarly appropriate and, what is even more important, no other approach offers the same combinations of analytic precision with geometrical descriptiveness. In electromagnetism the physicist will always need to understand and visualise how *electric* fields relate to *charges* and *magnetic* fields to *currents* – as vector theory tells him – even though he knows that from a more advanced point of view he should describe these fields combined as a tensor and the charge and current densities together as a 'vector in four-dimensional space–time'.

Contents

1 Summary of vector algebra

1 Addition and multiplication of vectors

The reader of this book is expected to be acquainted with the idea of representing a directed quantity (e.g. a displacement, velocity or force) by a vector $\boldsymbol{a}$, such that the associated length is denoted by

$$a = |\boldsymbol{a}| \tag{1}$$

and the direction by

$$\boldsymbol{n} \text{ (or } \hat{\boldsymbol{a}}) = \boldsymbol{a}/a. \tag{2}$$

He will also know the meaning of the sum and the difference of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

$$\boldsymbol{c} = \boldsymbol{a} \pm \boldsymbol{b} \tag{3}$$

and of the product of a vector $\boldsymbol{a}$ with a scalar† α,

$$\boldsymbol{b} = \alpha\boldsymbol{a}. \tag{4}$$

(Note that (2) is a special case of (4).)

In addition to the product defined by (4), vector algebra recognises two types of product formed from two (and only two) vectors. Firstly, let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two arbitrary vectors and let θ be the angle between their directions. Then the *scalar* product $\boldsymbol{a} \cdot \boldsymbol{b}$ of the two vectors is defined by

$$\boldsymbol{a} \cdot \boldsymbol{b} = ab \cos\theta. \tag{5}$$

Geometrically $\boldsymbol{a} \cdot \boldsymbol{b}$ is the projection of either of the vectors on to the direction of the other multiplied by the length of the latter. One notes that

$$\boldsymbol{a} \cdot \boldsymbol{a} = a^2$$

† A scalar may be simply a numerical factor or it may represent a physical quantity (such as mass, density or temperature) which can be defined *without reference to orientation in space*.

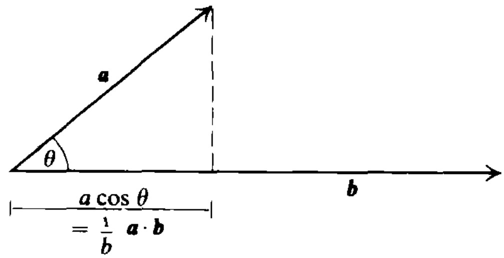

Fig. 1. The scalar product $\boldsymbol{a}\cdot\boldsymbol{b}$ of two vectors.

and that

$$a\cdot b = 0 \quad \text{if } a \text{ and } b \text{ are perpendicular.}$$

The introduction of this product enables one to write down identities like

$$|a \pm b|^2 = a^2 \pm 2a\cdot b + b^2 \tag{6}$$

closely resembling theorems of ordinary algebra. The scalar multiplication law is commutative,

$$a\cdot b = b\cdot a \tag{7}$$

and a distributive law

$$a\cdot(b+c) = a\cdot b + a\cdot c \tag{8}$$

is valid. Both (7) and (8) are used in stating (6).

As scalar multiplication is definable for two vectors and not more, it cannot be involved alone in any form of associative law. However, one may note that

$$\alpha(a\cdot b) = (\alpha a)\cdot b = a\cdot(\alpha b), \tag{9}$$

where α is a scalar, but that e.g.

$$a(b\cdot c) \neq (a\cdot b)c.$$

The second form of product of two vectors is the *vector product*.† Its definition is somewhat more complicated: The vector product $a \times b$ is the vector c that has the magnitude

$$c = ab \sin\theta \tag{10}$$

and the direction perpendicular to both a and b so that

† We are not here concerned with the fact that the possibility of defining a vector in this way is confined entirely to the physically interesting case of *three* space dimensions and even then requires a *conventional* choice of the direction into which the product vector shall point. From a more advanced point of view $a \times b$ is what is known as an axial vector or pseudo-vector. It is more properly related to the plane of the vectors a and b than to the direction perpendicular to it.

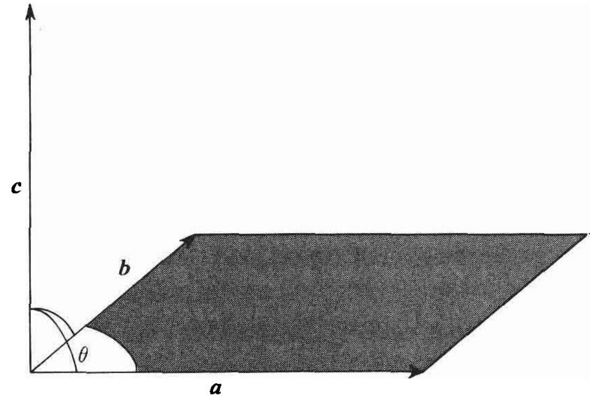

Fig. 2. The vector product $a \times b$ of two vectors.

$$c \cdot a = c \cdot b = 0. \tag{11}$$

Furthermore

a, b, c, *in that order*, form a right-handed system. (12)

The most striking and unusual feature of this definition of a product is that it is anticommutative and not commutative,

$$b \times a = -a \times b. \tag{13}$$

This follows from (12) since $b, a, -c$ and not b, a, c form a right-handed system. Either (10) or (13) demonstrates that

$$a \times a = 0. \tag{14}$$

Of particular importance to us is the fact that the magnitude c of $a \times b$ (see (10)) is the area of the parallelogram subtended by the two vectors a and b.

We refrain from giving the non-trivial proof that vector multiplication is distributive, i.e. that

$$a \times (b + c) = a \times b + a \times c \tag{15}$$

but would observe that the particular definition chosen for this vector product is determined by the need to have a law like (15). Without it this product construct would be of little use.

On the associative law the same remarks apply as for the scalar product. The identities

$$\alpha(a \times b) = (\alpha a) \times b = a \times (\alpha b) \tag{16}$$

are correct.

In addition to the two types of product formed from *two* vectors, it is useful to single out from among various products that are definable as a consequence of the rules already stated, one product involving *three* vectors, a, b and c. It is a *scalar*† product and apart from the sign it is the only scalar that can be formed from three vectors. It is

$$(a, b, c) = a\cdot(b \times c) = b\cdot(c \times a) = c\cdot(a \times b). \tag{17}$$

By virtue of (13) one sees that a non-cyclic interchange among the three vectors reverses the sign of this scalar triple product. If a, b and c form a right-handed triplet (and, in particular are not coplanar) the triple product (16) is positive and equal to the volume of the parallelepiped subtended by the three vectors. In general (using the modulus sign in its non-vectorial algebraic sense) that volume must be stated as $|(a, b, c)|$.

There are numerous identities that follow from the definitions given above. Without any proofs we provide a list of those that will be of use later in the book (a, b, c, d, e represent arbitrary vectors throughout):

$$a \times (b \times c) = (a\cdot c)b - (a\cdot b)c \tag{18}$$

$$a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0 \tag{19}$$

$$(a \times b)\cdot(c \times d) = (a\cdot c)(b\cdot d) - (a\cdot d)(b\cdot c). \tag{20}$$

For ease of reference at a particular point in the text below, it is worth re-writing the right hand side of (20) in a slightly different and rather clumsy looking form:

$$(a \times b)\cdot(c \times d) = b\cdot[(a\cdot c)d] - a\cdot[(b\cdot c)d]. \tag{20'}$$

This reformulation involves nothing more than a simple use of relation (9).

Two further identities of considerable importance are:

$$(a, b, c)d = (a\cdot d)(b \times c) + (b\cdot d)(c \times a) + (c\cdot d)(a \times b) \tag{21}$$

and a closely related equation:

$$(a, b, c)d = [(b \times c)\cdot d]a + [(c \times a)\cdot d]b + [(a \times b)\cdot d]c. \tag{22}$$

Finally, by multiplication of both sides of (21), by e, one finds

† In fact a pseudo-scalar; see footnote on p. 2.

$$(a, b, c)(d \cdot e) = (a \cdot d)[(b \times c) \cdot e] + (b \cdot d)[(c \times a) \cdot e] + (c \cdot d)[(a \times b) \cdot e]. \quad (23)$$

As for (20), a trivial re-formulation of (22) will be stated for future reference:

$$(a, b, c)(d \cdot e) = (b \times c) \cdot [(a \cdot d)e] + (c \times a) \cdot [(b \cdot d)e] + (a \times b) \cdot [(c \cdot d)e]. \quad (23')$$

The fact that one is concerned with space of precisely *three* dimensions is expressed mathematically by the statement that three, but not more than three, vectors can be linearly independent. Linear independence of three vectors means geometrically that (e.g. when drawn from the same origin) they do not lie in the same plane. Hence

$$(a, b, c) \neq 0 \quad (24)$$

means that a, b, c are linearly independent. If (24) holds, then any further vector d must be expressible as a linear combination of the first three. This is precisely what equation (22) puts in evidence.

One is not frequently interested in using (22) in full generality, but a special case of (22) is very well known indeed; this will be the subject of the next section.

2 *The cartesian components of a vector*

We introduce for the first time the notion of a set of three orthogonal unit vectors i, j and k, which in particular we choose (in that order) to be a right-handed set. Then we have

$$|i| = |j| = |k| = 1, \quad j \cdot k = k \cdot i = i \cdot j = 0, \quad (i, j, k) = +1. \quad (25)$$

The theorem that a general vector d may be represented as

$$d = d_1 i + d_2 j + d_3 k \quad (26)$$

is recognised at the very beginning of vector algebra and it is equally well known that

$$d_1 = d \cdot i, \quad d_2 = d \cdot j, \quad d_3 = d \cdot k. \quad (27)$$

The reader may care to check that (26) follows from the much more general (22) with the use of (25) and (27).

Having by (27) represented a general vector by its three 'cartesian'