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Marcus du Sautoy
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Zeta Functions of Groups and Rings

$\zeta_G(s)$

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To our families

Preface

The study of the subgroup growth of infinite groups is an area of mathematical research that has grown rapidly since its inception at the Groups St. Andrews conference in 1985. It has become a rich theory requiring tools from and having applications to many areas of group theory. Indeed, much of this progress is chronicled by Lubotzky and Segal within their book [42].

However, one area within this study has grown explosively in the last few years. This is the study of the zeta functions of groups with polynomial subgroup growth, in particular for torsion-free finitely-generated nilpotent groups. These zeta functions were introduced in [32], and other key papers in the development of this subject include [10, 17], with [19, 23, 15] as well as [42] presenting surveys of the area.

The purpose of this book is to bring into print significant and as yet unpublished work from three areas of the theory of zeta functions of groups.

First, there are now numerous calculations of zeta functions of groups by doctoral students of the first author which are yet to be made into printed form outside their theses. These explicit calculations provide evidence in favour of conjectures, or indeed can form inspiration and evidence for new conjectures. We record these zeta functions in Chap. 2. In particular, we document the functional equations frequently satisfied by the local factors. Explaining this phenomenon is, according to the first author and Segal [23], "one of the most intriguing open problems in the area".

A significant discovery made by the second author was a group where all but perhaps finitely many of the local zeta functions counting normal subgroups do not possess such a functional equation. Prior to this discovery, it was expected that all zeta functions of groups should satisfy a functional equations. Prompted by this counterexample, the second author has outlined a conjecture which offers a substantial demystification of this phenomenon. This conjecture and its ramifications are discussed in Chap. 4.

Finally, it was announced in [16] that the zeta functions of algebraic groups of types B_l, C_l and D_l all possessed a natural boundary, but this work is also yet to be made into print. In Chap. 5 we present a theory of natural boundaries of two-variable polynomials. This is followed by Chap. 6 where the aforementioned result on the zeta functions of classical groups is proved, and Chap. 7, where we consider the natural boundaries of the zeta functions attached to nilpotent groups listed in Chap. 2.

The first author thanks Zeev Rudnick who first informed him of Conjecture 1.11, Roger Heath-Brown who started the ball rolling and Fritz Grunewald for discussions which helped bring the ball to a stop. The first author also thanks the Max-Planck Institute in Bonn for hospitality during the preparation of this work and the Royal Society for support in the form of a University Research Fellowship. The second author thanks the EPSRC for a Research Studentship and a Postdoctoral Research Fellowship, and the first author for supervision during his doctoral studies.

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Contents

1

Introduction

1.1 A Brief History of Zeta Functions

Zeta functions are analytic functions with remarkable properties. They have played a crucial role in the proof of many significant theorems in mathematics: Dirichlet's theorem on primes in arithmetic progressions, the Prime Number Theorem, and the proofs of the Weil conjectures and the Taniyama–Shimura conjecture to name just a few.

Many different types of zeta function have been defined. We summarise below some of the more significant ones.

1.1.1 Euler, Riemann

In the eighteenth century a number of mathematicians were interested in determining the precise value of the infinite series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots , \tag{1.1}$$

the sum of the squares of the harmonic series. Daniel Bernoulli suggested 8/5 as an estimate for its value, but it was Leonhard Euler who first gave the precise value of this sum. To do this, Euler defined the *zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

for $s \in \mathbb{R}$, $s > 1$. The infinite sum (1.1) is then the zeta function evaluated at $s = 2$. However Euler was able to do more than just give the value of $\zeta(2)$. He gave a formula for the zeta function at every even positive integer:

$$\zeta(2m) = \frac{2^{2m-1}\pi^{2m}|B_{2m}|}{(2m)!} .$$

As an acknowledgement of the support the Bernoulli family had given him, he was able to identify the rational constants B_{2m} as the Bernoulli numbers discovered by Daniel's uncle Jacob. Since $B_2 = 1/12$, it follows that $\zeta(2) = \pi^2/6$. To this day, nobody has been able to find a comparable expression for the zeta function at odd integers. It is not even known if $\zeta(3)$ is transcendental.

Euler also discovered the *Euler product identity*. If one sets

$$\zeta_p(s) = \sum_{n=0}^{\infty} p^{-ns} = \frac{1}{1-p^{-s}} ,$$

then

$$\zeta(s) = \prod_p \zeta_p(s) ,$$

where the product is over all primes p. This identity is fundamental to the connection between the zeta function and the primes. As well as encapsulating the Fundamental Theorem of Arithmetic, it also offers a simple analytic proof of a classical result on primes: the fact that the harmonic series $1+1/2+\cdots+1/n+\cdots$ diverges means that there must be infinitely many primes.

The zeta function converges for $s > 1$ but diverges at $s = 1$. Later, Bernhard Riemann, inspired by Cauchy's work on functions of a complex variable, considered the zeta function as a function on $\mathbb{C}$. By doing so, he could analytically continue the zeta function around the pole at $s = 1$, and obtain a function meromorphic on the whole complex plane. The pole at $s = 1$ is simple and is the only singularity of the zeta function. Furthermore, Riemann showed that this zeta function satisfies a *functional equation*. If one sets $\xi(s) = \Gamma(s/2)\pi^{-s/2}\zeta(s)$, where $\Gamma(s)$ is the gamma function, then

$$\xi(s) = \xi(1-s) . \tag{1.2}$$

This analytically-continued function is now known as the *Riemann zeta function* in honour of Riemann's achievements with it.

Since the zeta function is nonzero for $\Re(s) \geq 1$, the only zeros of the Riemann zeta function with $\Re(s) \leq 0$ are the trivial zeros at negative even integers. Hence the only other zeros are those within the *critical strip*, $0 < \Re(s) < 1$. Riemann famously hypothesised that all the zeros lie on the *critical line* $\Re(s) = \frac{1}{2}$. Hardy and Littlewood [33] have since proved the existence of infinitely many zeros on the critical line and Conrey [3] has proved that more than 40% of the zeros lie on the line. At the time of writing, the most recent computer calculation [27] seems to have confirmed that the first ten trillion (10^{13}) Riemann zeros are on the line. Despite all this evidence, it is still not known whether a zero lies off the line.

Such is the importance of this Hypothesis that there is a considerable body of mathematical work which depends on the truth of this Hypothesis. Its proof would simultaneously prove numerous other theorems for which its truth has had to be assumed. Furthermore, its status as one of the Clay Mathematics Institute Millennium Prize Problems would also earn its author a million-dollar prize.

Hadamard and de la Vallée Poussin were also able to utilise the power of the Riemann zeta function. By showing that the Riemann zeta function is nonzero on $\Re(s) = 1$, they independently proved the Prime Number Theorem, that

$$\lim_{n\to\infty} \frac{\pi(n)\log n}{n} = 1 ,$$

where $\pi(n)$ is the number of primes no larger than n.

1.1.2 Dirichlet

In the meantime, Dirichlet was taking the concept of the zeta function in a new direction. His major innovation was to attach a coefficient a_n to each term n^{-s}. Recall that the Riemann zeta function is defined for $\Re(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} .$$

A Dirichlet character with period m is a function $\chi : \mathbb{N}_{>0} \to \mathbb{C}$ that has the following properties:

- χ is totally multiplicative, i.e. $\chi(1) = 1$ and $\chi(n_1)\chi(n_2) = \chi(n_1 n_2)$ for all $n_1, n_2 \in \mathbb{N}_{>0}$.
- $\chi(m+n) = \chi(n)$ for all $n \in \mathbb{N}_{>0}$.
- $\chi(n) = 0$ if $\gcd(n, m) > 1$.

The *Dirichlet L-function* of χ is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} .$$

Using these L-functions, Dirichlet proved that if $\gcd(r, N) = 1$, the arithmetic progression r, $r + N$, $r + 2N$, ... contains infinitely many primes. Furthermore, his proof yields the additional result that the primes are in some sense evenly distributed amongst the congruence classes of integers coprime to N. In honour of this achievement, any function of the form $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is called a Dirichlet series.

If $m = 1$ then χ is the trivial character, hence $L(s,\chi) = \zeta(s)$, the Riemann zeta function once again, which we know can be meromorphically continued to $\mathbb{C}$. If $m > 1$, $L(s,\chi)$ can be analytically continued to an entire function on $\mathbb{C}$. Indeed, the fact that $L(s,\chi)$ is nonzero at $s = 1$ for nontrivial characters χ plays a key part in Dirichlet's proof. A functional equation of $L(s,\chi)$ which takes a similar shape to (1.2) can also be given, however its statement is less succinct than that satisfied by the Riemann zeta function. We refer the interested reader to the section on Dirichlet L-functions in [37].

The multiplicativity of the characters χ leads easily to an Euler product for the Dirichlet L-function,

$$L(s,\chi) = \prod_p \frac{1}{1-\chi(p)p^{-s}} .$$

Indeed, it is easy to see that any Dirichlet series where the sequence (a_n) grows at most polynomially in n and is totally multiplicative (i.e. $a_m a_n = a_{mn}$ for all $m, n \in \mathbb{N}$) satisfies such an Euler product.

1.1.3 Dedekind

The zeta functions described above have had predominantly number-theoretic applications. It was Dedekind who was perhaps the first to use zeta functions for an algebraic purpose. For K a finite extension of the rational numbers $\mathbb{Q}$, the *Dedekind zeta function* of the field K is defined by

$$\zeta_K(s) = \sum_{\mathfrak{a}} |\vartheta_K : \mathfrak{a}|^{-s} ,$$

where $|\vartheta_K : \mathfrak{a}|$ is the index of the ideal $\mathfrak{a}$ in the ring of integers ϑ_K and the sum is over all nonzero ideals $\mathfrak{a}$ in ϑ_K. Again, this zeta function extends to a meromorphic function on $\mathbb{C}$, with a simple pole at $s = 1$.

Perhaps one of the most remarkable properties of the Dedekind zeta function is the *class number formula*, which encodes the class number of the field in the residue of the pole of $\zeta_K(s)$ at $s = 1$. If $\Delta(K)$ is the discriminant of the field K, R_K the *regulator* of K, u the order of the group of roots of unity within the ring of integers ϑ_K, r_1 (resp. r_2) is the number of real (resp. the number of pairs of complex conjugate) embeddings of K and h_K the class-number of K, then

$$\mathrm{Res}_{s=1}(\zeta_K(s)) = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{u\sqrt{|\Delta(K)|}} .$$

As with the Riemann zeta function and Dirichlet L-functions, the Dedekind zeta function satisfies a functional equation. Let $n = |K : \mathbb{Q}|$, the degree of the field extension, and put

$$\Xi_K(s) = \left(\frac{\sqrt{|\Delta(K)|}}{2^{r_2}\pi^{n/2}}\right)^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2}\zeta_K(s) .$$

Then $\Xi_K(s) = \Xi_K(1-s)$.

1.1.4 Artin, Weil

Dedekind's zeta function considers finite extensions of the rational numbers $\mathbb{Q}$. E. Artin considered zeta functions connected to finite extensions of global fields of characteristic p. One particular example he considered was the field $K = \mathbb{F}_p(x)(\sqrt{x^3 - x})$, i.e. the field of rational functions with coefficients in $\mathbb{F}_p(x)$ extended by adjoining $\sqrt{x^3 - x}$. Let R be the integral closure of $\mathbb{F}_p[x]$ in K. Artin considered the zeta function

$$\zeta_R(s) = \sum_{\mathfrak{a} \trianglelefteq R} |R : \mathfrak{a}|^{-s} .$$

If one sets $y = \sqrt{x^3 - x}$, then quite clearly we have an elliptic curve $y^2 = x^3 - x$. Artin found that the zeta function $\zeta_R(s)$ was encoding the number of points on this elliptic curve. In particular,

$$\zeta_R(s) = (1 - p^{-s}) \exp\left(\sum_{m=1}^{\infty} \frac{N_{p^m} p^{-ms}}{m} \right) ,$$

where

$$N_{p^m} = |\{ (a, b) \in \mathbb{F}_{p^m}^2 : b^2 = a^3 - a \}| + 1 .$$

The extra term is necessary to count the point at infinity in projective space. Furthermore, Artin could show, for this elliptic curve and about 40 others, that

$$\exp\left(\sum_{m=1}^{\infty} \frac{N_{p^m} p^{-ms}}{m} \right) = \frac{(1 + \pi_p p^{-s})(1 + \bar{\pi}_p p^{-s})}{(1 - p^{-s})(1 - p^{1-s})}$$

for a certain pair of complex conjugate numbers π_p and $\bar{\pi}_p$ which depend on the elliptic curve. Hasse later extended this result to all elliptic curves, and Weil to all smooth projective curves of arbitrary genus. Indeed, this property that the zeros of the zeta function satisfy $|\pi| = p^{1/2}$ is known as the *analogue of the Riemann Hypothesis* for the zeta function.

Weil was inspired by his work to consider the zeta function of an arbitrary smooth projective variety X defined over a finite field $\mathbb{F}_q$. This is defined analogously to Artin's zeta function, but omitting the factor $(1 - p^{-s})$, by

$$\zeta_X(s) = \exp\left(\sum_{m=1}^{\infty} \frac{N_{q^m} q^{-ms}}{m} \right) ,$$

where N_{q^m} is the number of points on X over the field $\mathbb{F}_{q^m}$. In particular, $\zeta_X(s)$ was conjectured to always be a rational function in q^{-s}, and to satisfy the functional equation $\zeta_X(n - s) = \pm q^{(\frac{1}{2}n-s)C} \zeta_X(s)$, for some constant C which can be given explicitly in terms of geometrical invariants of X. Weil was also able to formulate a strategy for proving these conjectures. He observed that if one has a suitable cohomology theory similar to that for varieties defined over $\mathbb{C}$, the conjectures follow from various standard properties of this cohomology theory. This observation motivated the development of various cohomology theories and eventually led to the development of the l-adic cohomology by Grothendieck and M. Artin, successfully employed by Deligne to confirm these conjectures.

1.1.5 Birch, Swinnerton-Dyer

If one has a polynomial equation over $\mathbb{Z}$, one can reduce it modulo p to give a variety defined over a finite field. So, given the zeta functions for the reductions mod p, what do we get when we multiply them all together? Does this 'global' zeta function tell us anything about the solutions of the original polynomial over $\mathbb{Q}$ or $\mathbb{Z}$?

In the case where X is an elliptic curve defined over $\mathbb{Q}$, such a global zeta function has been defined. If E is an elliptic curve over $\mathbb{Q}$, the L-function of E is defined by[1]

$$L(E,s) = \prod_{p \nmid 2\Delta} \frac{1}{1 - a_p p^{-s} + p^{1-2s}} ,$$

where Δ is the discriminant of E, N_p is the number of points on E mod p and $a_p = p - N_p$. This Dirichlet series converges for $\Re(s) > \frac{3}{2}$ and thanks to the complete proof of the Taniyama–Shimura conjecture [1], it is known that $L(E,s)$ can be analytically continued to an entire function. A functional equation relating $L(E,s)$ and $L(E,2-s)$ also follows from Taniyama–Shimura. It was conjectured by Birch and Swinnerton-Dyer that E has infinitely many rational points if and only if $L(E,s)$ is zero at $s = 1$, and furthermore the torsion-free rank of the Mordell–Weil group of points on E over $\mathbb{Q}$ is the order of the zero at $s = 1$. Coates and Wiles [2] have proved that if $L(E,1) \neq 0$ then E has only finitely many rational points, and it has since been shown that the conjecture is true for $r \leq 1$ [5]. However the rest of the conjecture remains open. Like the Riemann Hypothesis, the Clay Foundation offers a million-dollar prize for the proof of this conjecture.

1.2 Zeta Functions of Groups

By no means is the above a complete list of zeta functions. We have omitted more than we have included, for we simply do not have the space to list them all. The final chapter of the Encyclopedic Dictionary of Mathematics [37] is

[1] There are factors associated to the primes $p \mid 2\Delta$ but for simplicity we ignore them.

a good place to start for those keen to know more about the panoply of zeta functions.

Furthermore, the Encyclopedic Dictionary also lists four basic properties a zeta function should ideally satisfy:

(ZF1) It should be meromorphic on the whole complex plane
(ZF2) It should have a Dirichlet series expansion
(ZF3) There should be some natural Euler product expansion
(ZF4) It should satisfy a functional equation

All the zeta functions we listed above satisfy all four of these properties. It may also be of interest to determine the residue of the zeta function at a pole, whenever such a singularity exists.

In this book, we consider these criteria for a relative newcomer to the family of zeta functions, zeta functions of groups and rings. We cannot expect that these zeta functions will reach the same lofty heights as the zeta functions presented above, but we do hope the reader agrees with our viewpoint that there is interesting mathematics concerning zeta functions of groups.

1.2.1 Zeta Functions of Algebraic Groups

The first example of a zeta function of a group is associated to a $\mathbb{Q}$-algebraic group $\mathfrak{G}$ with a choice of some $\mathbb{Q}$-rational representation $\rho : \mathfrak{G} \to \mathrm{GL}_n$. The zeta function $Z_{\mathfrak{G},\rho}(s)$ of $\mathfrak{G}$ has been defined as the Euler product over all primes p of the following local zeta functions defined by p-adic integrals with respect to the normalised Haar measure $\mu_{\mathfrak{G}}$ on $\mathfrak{G}(\mathbb{Z}_p)$:

$$Z_{\mathfrak{G},\rho,p}(s) = \int_{\mathfrak{G}_p^+} |\det(\rho(g))|_p^s \,\mathrm{d}\mu_{\mathfrak{G}}(g) ,$$

where $\mathfrak{G}_p^+ = \rho^{-1}(\rho(\mathfrak{G}(\mathbb{Q}_p)) \cap \mathrm{M}_n(\mathbb{Z}_p))$ and $|\cdot|_p$ denotes the p-adic norm.

The definition of the zeta function of an algebraic group goes back to the work of Hey [35] who recognised that the zeta function attached to the algebraic group GL_n could be used to encode the subalgebra structure of central simple algebras. In the 1960s, Tamagawa established in [56] the meromorphic continuation of the zeta functions of Hey attached to GL_n. Subsequently, Satake [50] and Macdonald [43] considered zeta functions of other reductive groups. But it is the work of Igusa [36] in the 1980s that established explicit expressions for the local factors of Chevalley groups which allow for some analysis of the analytic behaviour of the global zeta functions. In particular his work shows that the zeta function is built from Riemann zeta functions and functions of the form

$$Z(s) = \prod_{p \text{ prime}} W(p, p^{-s}) , \tag{1.3}$$