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James F. Peters · Andrzej Skowron
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James F. Peters Andrzej Skowron (Eds.)

Transactions on Rough Sets IV

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Preface

Volume IV of the Transactions on Rough Sets (TRS) introduces a number of new advances in the theory and application of rough sets. Rough sets and approximation spaces were introduced more than 30 years ago by Zdzisław Pawlak. These advances have profound implications in a number of research areas such as the foundations of rough sets, approximate reasoning, artificial intelligence, bioinformatics, computational intelligence, cognitive science, intelligent systems, data mining, machine intelligence, and security. In addition, it is evident from the papers included in this volume that the foundations and applications of rough sets is a very active research area worldwide. A total of 16 researchers from 7 countries are represented in this volume, namely, Canada, India, Norway, Sweden, Poland, Russia and the United States of America. Evidence of the vigor, breadth and depth of research in the theory and applications of rough sets can be found in the 10 articles in this volume.

Prof. Pawlak has contributed a treatise on the philosophical underpinnings of rough sets. In this treatise, observations are made about the Cantor notion of a set, antinomies arising from Cantor sets, the problem of vagueness (especially, *vague* (*imprecise*) concepts), fuzzy sets, rough sets, fuzzy vs. rough sets as well as logic and rough sets. Among the many vistas and research directions suggested by Prof. Pawlak, one of the most fruitful concerns the model for a rough membership function, which was incarnated in many different forms since its introduction by Pawlak and Skowron in 1994. Recall, here, that Prof. Pawlak introduced approximation spaces in the context of rough sets during the early 1980s. Later, the model for rough membership provided a basis for a model for rough inclusion in generalized approximation spaces introduced by Skowron and Stepaniuk during the early 1990s.

In addition, this volume includes seven papers that explore the theory of rough sets, and two papers that present new applications of rough sets. New developments in rough set theory are represented by papers that investigate a framework for reasoning with rough sets utilizing extended logic programs (Aida Vitória), optimization of decision trees (Igor V. Chikalov, Mikhail Ju. Moshkov, and Maria S. Zelentsova), fuzzy set and rough set approaches to dealing with missing data (Dan Li, Jitender Deogun, William Spaulding, and Bill Shuart), generalization of the indiscernibility relation as an aid to dealing with incompletely specified decision tables (Jerzy W. Grzymała-Busse), deterministic and non-deterministic decision tree complexity in the context of both finite and infinite information systems (Mikhail Ju. Moshkov), analogy-based reasoning in classifier construction (Arkadiusz Wojna), and incremental learning and evaluation of structures of rough decision tables (Wojciech Ziarko). In addition, two papers in this volume introduce new applications of rough sets, namely, supervised learning in the gene ontology (Herman Midelfart) and the design of an intrusion detection system (Sanjay Rawat, V.P. Gulati, and Arun K. Pujari).

This issue of the TRS was made possible thanks to the laudable efforts of a great many generous persons and organizations. We express our thanks to the many anonymous reviewers for their heroic efforts in providing detailed reviews of the articles in this issue of the TRS. The editors and authors of this volume also extend an expression of gratitude to Alfred Hofmann, Ursula Barth, Christine Günther and the LNCS staff at Springer for their support in making this volume of the TRS possible. In addition, the editors of this volume extend their thanks to Marcin Szczuka for his consummate skill and care in the compilation of this volume. The editors of this volume have been supported by the Ministry for Scientific Research and Information Technology of the Republic of Poland, Research Grant No. 3T11C00226, and the Natural Sciences and Engineering Research Council of Canada (NSERC) Research Grant 185986.

July 2005

James F. Peters
Andrzej Skowron

LNCS Transactions on Rough Sets

This journal subline has as its principal aim the fostering of professional exchanges between scientists and practitioners who are interested in the foundations and applications of rough sets. Topics include foundations and applications of rough sets as well as foundations and applications of hybrid methods combining rough sets with other approaches important for the development of intelligent systems.

The journal includes high-quality research articles accepted for publication on the basis of thorough peer reviews. Dissertations and monographs up to 250 pages that include new research results can also be considered as regular papers. Extended and revised versions of selected papers from conferences can also be included in regular or special issues of the journal.

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A Treatise on Rough Sets

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The central problem of our age is how to act decisively in the absence of certainty.
Bertrand Russell (1950). An Inquiry into Meaning and Truth.
George Allen and Unwin, London;
W.W. Norton, New York

Abstract. This article presents some general remarks on rough sets and their place in general picture of research on vagueness and uncertainty - concepts of utmost interest, for many years, for philosophers, mathematicians, logicians and recently also for computer scientists and engineers particularly those working in such areas as AI, computational intelligence, intelligent systems, cognitive science, data mining and machine learning. Thus this article is intended to present some philosophical observations rather than to consider technical details or applications of rough set theory. Therefore we also refrain from presentation of many interesting applications and some generalizations of the theory.

Keywords: Sets, fuzzy sets, rough sets, antinomies, vagueness.

1 Introduction

In this article we are going to give some general remarks on rough sets and their place in general picture of research on vagueness and uncertainty - concepts of utmost interest, for many years, for philosophers, mathematicians, logicians and recently also for computer scientists and engineers particularly those working in such areas as AI, computational intelligence, intelligent systems, cognitive science, data mining and machine learning. Thus this article is intended to present some philosophical observations rather than to consider technical details or applications of rough set theory. Therefore we also refrain from presentation of many interesting applications and some generalizations of the theory.

We start our consideration in Section 2 with general comments on classical notion of a set, formulated by Georg Cantor [8] over one hundred years ago. Next, we discuss briefly in Section 3 a source of basic discomfort of classical set theory, namely the antinomies, which shocked the foundation of mathematics

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and the ways out of this embarrassment. Further, the notion of vagueness and its role in mathematics, as formulated by Gottlob Frege [12] are briefly discussed in Section 4. The basic notions concerning fuzzy sets and rough sets are presented in Sections 5 and 6, respectively. The contrast between fuzzy membership [51] and rough membership [31] is briefly considered in Section 7. Than we discuss the notions of fuzzy set [51] and rough set [28,30] as certain formalizations of vagueness. A brief comparison of both notions close this section.

We conclude our deliberation in Section 8 with brief discussion of deductive, inductive and common sense reasoning and the role of rough sets has played in these kinds of inference.

2 Sets

The notion of a set is the basic one of mathematics. All mathematical structures refer to it.

The definition of this notion and the creation of set theory are due to German mathematician Georg Cantor (1845-1918), who laid the foundations of contemporary set theory about 100 years ago. The original, intuitive definition of the Cantor's notion of the set [8] is given below:

> *"Unter einer "Manningfaltigkeit" oder "Menge" verstehe ich nämlich allgemein jedes Viele, welches sich als Eines denken lässt, d.h. jeden Inbegriff bestimmter Elemente, welcher durch ein Gesetz zu einem Ganzen verbunden werden kann."*

Thus according to Cantor a set is a collection of any objects, which according to some law can be considered as a whole. As one can see the notion is very intuitive and simple.

All mathematical objects, e.g., relations, functions, numbers, etc. are some kind of sets. In fact set theory is needed in mathematics to provide rigor.

The notion of a set is not only fundamental for the whole mathematics but it also plays an important role in natural language. We often speak about sets (collections) of various objects of interest such as, collection of books, paintings and people.

The intuitive meaning of a set according to some dictionaries is the following:

"A number of things of the same kind that belong or are used together."

Webster's Dictionary

"Number of things of the same kind, that belong together because they are similar or complementary to each other."

The Oxford English Dictionary

Thus a set is a collection of things which are somehow related to each other but the nature of this relationship is not specified in these definitions.

In fact, these definitions are due to the original definition given by Cantor.

3 Antinomies

Well! I'have seen often a cat without a grin, thought Alice; but a grin without a cat!
Lewis Carroll (1994). Alice's Adventures in Wonderland.
Penguin Books, London

In 1903 the renown English philosopher Bertrand Russell (1872-1970) observed [37] that the intuitive notion of a set given by Cantor leads to logical *antinomies* (contradictions), i.e., set theory was contradictory (there also exist other kinds of antinomies - we refrain from considering them here). A logical antinomy, for the sake of simplicity called antinomy in the remaining part of this paper, arises when after carrying on a correct logical reasoning we come to a contradiction, i.e., to the propositions A and *non-A*, which is not allowed in logic.

As an example let us discuss briefly the so-called Russell's antinomy. Consider the set X containing all the sets Y, which are not the elements of themselves. If we assume that X is its own element then X, by definition, cannot be its own element; while if we assume that X is not its own element then, according to the definition of the set X, it must be its own element. Thus while applying each assumption we obtain contradiction.

The above antinomy is often illustrated with the example of a barber, who got the instruction, that he could only shave all the men who did not shave themselves. Then a question arises if he may shave himself or not. If we assume that the barber shaves himself then, according to the instruction, he may not shave himself. But when we assume that he does not shave himself then, according to the instruction, he should shave himself. Thus we have run across an antinomy.

Another well known antinomy, called the power-set antinomy, goes as follows: consider (infinite) set X of all sets. Thus X is the greatest set. Let Y denote the set of all subsets of X. Obviously Y is greater then X, because the cardinality of the family of all subsets of a given set is always greater the cardinality of the set of all its elements. For example, if $X = \{1, 2, 3\}$ then $Y = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$, where $\emptyset$ denotes the empty set. Hence, X is not the greatest set as assumed and we have arrived at contradiction.

Antinomies show that a set cannot be a collection of arbitrary elements, as was stipulated by Cantor.

One could think that antinomies are ingenuous logical play, but it is not so. They question the essence of logical reasoning. That is why there have been attempts to "repair" Cantor's theory for over 100 years or to substitute another set theory for it but the results have not been good so far. Is then all mathematics based on doubtful foundations?

As a remedy for this defect several improvements of set theory have been proposed. For example,

- Axiomatic set theory (Zermello and Fraenkel, 1904);
- Theory of types (Whitehead and Russell, 1910);
- Theory of classes (v. Neumann, 1920).

All of these improvements consist in restrictions put on objects which can form a set. Such restrictions are expressed by properly chosen axioms, which say how a set can be built. They are called, in contrast to Cantors' intuitive set theory, axiomatic set theories.

Instead of improvements of Cantors' set theory by its axiomatization, some mathematicians proposed escape from classical set theory by creating a completely new idea of a set, which would free the theory from antinomies. Some of them are listed below.

- Mereology (Leśniewski, 1915, [19]);
- Alternative set theory (Vopenka, 1970, [49]);
- "Penumbral" set theory (Apostoli and Kanada, 1999, [1]).

No doubt the most interesting proposal was given by Polish logician Stanisław Leśniewski, who proposed instead of membership relation between elements and sets, employed in classical set theory, the relation of "being a part". In his set theory, called *mereology*, this relation is a fundamental one [19].

None of the three mentioned above "new" set theories were accepted by mathematicians. However, Leniewski's mereology attracted some attention of philosophers and recently also researchers in computer science (see, e.g., [9,33,43]).

The problem of finding an alternative to classical set theory has failed to be solved until now.

Basic concept of mathematics, the set, leads to antinomies, i.e., it is contradictory. How is it is then possible that mathematics is so successful and can be applied almost everywhere – that bridges are not collapsing, air-planes are not falling down and man has landed on the moon?

The deficiency of sets, mentioned above, has rather philosophical than practical meaning, since sets used practically in mathematics are free from the above discussed faults. Antinomies are associated with very "artificial" sets constructed in logic but not found in sets used in "everyday" mathematics. That is why we can use mathematics safely.

4 Vagueness

Besides known and unknown what else is three?
Harold Pinter (1965). The Homecoming.
Methuen, London

Another issue discussed in connection with the notion of a set is vagueness. Mathematics requires that all mathematical notions (including set) must be exact, otherwise precise reasoning would be impossible. However, philosophers [17,18,36,38] and recently computer scientists [21,23,24,41] as well as other researchers have become interested in *vague* (imprecise) concepts.

In classical set theory a set is uniquely determined by its elements. In other words, this means that every element must be uniquely classified as belonging to the set or not. That is to say the notion of a set is a *crisp* (precise) one. For example, the set of odd numbers is crisp because every number is either odd or even.

In contrast to odd numbers, the notion of a beautiful painting is vague, because we are unable to classify uniquely all paintings into two classes: beautiful and not beautiful. Some paintings cannot be decided whether they are beautiful or not and thus they remain in the doubtful area. Thus *beauty* is not a precise but a vague concept.

Almost all concepts we are using in natural language are vague. Therefore common sense reasoning based on natural language must be based on vague concepts and not on classical logic. This is why vagueness is important for philosophers and recently also for computer scientists. Interesting discussion of this issue can be found in [36].

The idea of vagueness can be traced back to ancient Greek philosophers Eubulides (ca. 400BC) who first formulated so called sorites (Bald Man or Heap) paradox. The paradox goes as follows: suppose a man has 100 000 hair on his head. Removing on hair from his head surely cannot make him bald. Repeating this step we arrive at the conclusion the a man without any hair is not bald. Similar reasoning can be applied to a hip stones.

Vagueness is usually associated with the boundary region approach (i.e., existence of objects which cannot be uniquely classified relative to a set or its complement) which was first formulated in 1893 by the father of modern logic, German logician, Gottlob Frege (1848-1925). He wrote:

"Der Begrieff muss scharf begrenzt sein. Einem unscharf begrenzten Begriff würde ein Bezirk ensprechen, der nicht überall ein scharfe Grentzlinie hätte, sondern stellenweise gantz verschwimmend in die Umgebung übergine" [12].

Thus according to Frege:

"The concept must have a sharp boundary. To the concept without a sharp boundary there would correspond an area that had not a sharp boundary-line all around."

It means mathematics must use crisp, not vague concepts, otherwise it would be impossible to reason precisely. Summing up, vagueness is

- not allowed in mathematics;
- interesting for philosophy;
- necessary for computer science.

5 Fuzzy Sets

There is nothing new under the sun.
Ecclesiates 1:9

At the same time, independently of mathematicians' and philosophers' investigations, engineers became interested in the notion of a set. It turned out that many practical problems could not be formulated and solved by means of classical Cantor's notion of a set.

In 1965 Lotfi Zadeh, Professor of University of Berkely, proposed a different notion of a set, in which elements can belong to a set to some extent and not definitively, as it is in case of the classical set theory. This proposal turned out applicable in many domains and initiated extensive research in fuzzy set theory, what became the name of Zadeh's theory [43].

In his approach an element can belong to a set to a degree k $(0 \leq k \leq 1)$, in contrast to classical set theory, where an element must definitely belong or not to a set. For example, in classical set theory one can say that someone is definitely ill or healthy, whereas in the fuzzy set theory language we can say that someone is ill (or healthy) at the 60 percent level (i.e., in degree 0.6).

Let us observe that the definition of fuzzy set involves more advanced mathematical concepts – real numbers and functions – whereas in classical set theory the notion of a set is used as a fundamental notion of whole mathematics and is used to derive any other mathematical concepts, e.g., numbers and functions. Consequently fuzzy set theory cannot replace classical set theory, because, in fact, the theory is needed to define fuzzy sets.

Fuzzy membership function has the following properties:

a) $\mu_{U-X}(x) = 1 - \mu_X(x)$ for any $x \in U$;
b) $\mu_{X \cup Y}(x) = max(\mu_X(x),\ \mu_Y(x))$ for any $x \in U$;
c) $\mu_{X \cap Y}(x) = min(\mu_X(x),\ \mu_Y(x))$ for any $x \in U$.

That means that the membership of an element to the union and intersection of sets is uniquely determined by its membership to constituent sets. This is a very nice property and allows very simple operations on fuzzy sets, which is a very important feature both theoretically and practically.

Several generalizations of this basic approach to concept approximation are presented in the literature (see, e.g., [14,42,44,45,50]).

Let us stress once more that classical set is a primitive notion and is defined intuitively or axiomatically. Fuzzy sets are defined by employing the fuzzy membership function, which involves advanced mathematical structures, numbers and functions. Thus it cannot play an analogous role in mathematics similar to that played by the classical concept of a set, which is used to define numbers and functions.

Fuzzy set theory can be perceived as new model of vagueness. The theory and its applications developed very extensively over the past four decades and attracted attention of engineers, logicians, mathematicians and philosophers worldwide.

6 Rough Sets

Data! data! data!
Sir Artur Conan Doyle (1994). The Adventures of Sherlock Holmes. Penguin Books, London

Rough set theory, proposed by the author in 1982 [28,30], is still another approach to vagueness.

Rough set theory expresses vagueness not by means of membership, but by employing a boundary region of a set. If the boundary region of a set is empty