Signals and Linear Systems

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Preface

When viewing the number of linear system texts available to students and teachers, any author adding to this collection must feel a need to justify his work. This text grew out of several editions of classroom notes which we developed for a senior-level course at the University of Colorado. While organizing the course, we found no single text which covered the material we thought important at a level suitable for undergraduate students. In particular, material on discrete-time systems was scattered, and most texts were either directed toward a particular type of system—such as electrical circuits, control systems, or communication systems—or were arranged in a topical order which the students found difficult to follow.

Our intent in writing this book is to present, in an organizational format designed for the student, the basic general techniques for analyzing linear systems. We treat both the usual continuous-time systems, and also discrete-time systems, which are finding widespread use in modern communication and control systems. The material is so ordered that the topics covered reinforce one another. The student is led naturally from basic techniques of time-domain analysis to the more abstract, although computationally simpler, transform-domain techniques. Extensive use is made of examples to illustrate the use of these techniques in solving problems in many diverse areas. Several examples are treated using two or more techniques in order to aid the student in comparing and relating the different solution methods. The material is general, and is chosen to lead, if desired, into a second-semester course in communication systems, control systems, or other areas which use these basic techniques in specific advanced applications.

Prerequisites assumed of the student are a sophomore-level mathematics course covering differential equations, and a course in the student's major area (such as electrical circuits or mechanics) which treats the derivation of mathematical models of physical systems. A course in probability theory is also helpful; examples and problems in the text which illustrate applications to probability problems are identified with the symbol * and may be omitted by those without this background.

This text is used for a one-semester elective course in linear system analysis at the University of Colorado. The course meets for a total of forty lecture hours; in this time we cover most of Chapters 1–6 and some of Chapter 7 if time permits. There is enough material in the text to give the instructor a choice of emphasis. The following notes indicate, as a guide, how we treat the various chapters.

Chapter 1 is covered rapidly, since for most students this is a review. The classical solution of linear constant coefficient differential equations, involving sums of functions of the type ce^{rt}, is readily extended to difference equation solutions, with sums of sequences of the type cr^k

Chapter 2 represents a departure from the usual order found in most linear systems texts. We cover the concept of convolution for both discrete-time and continuous-time systems in some depth. We feel that convolutional methods are so important that they should not be avoided by the use of transform methods. The convolutional sum for discrete-time systems and the corresponding convolutional integral for continuous-time systems furnish the student with a simple and effective way of picturing the operation of a linear system. We introduce convolutional methods in terms of discrete-time systems because the impulse sequence $\{\delta_k\}$ is easily defined and understood. After covering discrete-time systems we progress to continuous-time systems. In using convolutional methods for analysis of linear systems, the impulse response sequence or function of the system must be known. We present a method of calculating the impulse response sequence or function based directly on the difference or differential equation modeling the system, which is more general and usually simpler than the corresponding transform methods.

The state variable material of Chapter 3 is included for two reasons: (1) the larger, more complex systems now treated by control, power, and communication engineers require this compact general description; (2) computer solution methods are well-suited to the matrix manipulation involved. Again, we find that the solution of discrete-time problems is simpler and more direct, and hence it is treated first. The solution of continuous-time problems then follows quite naturally.

Chapter 4 is placed deliberately to introduce students to transform-domain solutions. Since students are probably least familiar with Z-transforms, they are not distracted by mechanical techniques (such as those they have used previously with Laplace transforms) and can concentrate on the central ideas involved in transforming from one domain to another. The analogous use of logarithms discussed in the chapter introduction is useful in this respect.

Chapter 5 introduces transform methods for continuous-time systems by considering the Fourier series and Fourier integral as methods of representing continuous-time functions. The generalized Fourier series is covered in depth in order to suggest why one might wish to have alternate representations of functions available. Walsh functions, for example, are shown to represent rectangular waveforms more "naturally" than the usual sinusoids. The Fourier transform is introduced as a generalization of the exponential Fourier series. The properties of the Fourier integral are introduced, pointing out the similarity with the Z-transform properties, and these properties are then used to obtain transforms of "energy signals" and then so-called "power signals." Power signals are time functions which are not absolutely integrable but do have finite power. By using the definition of the impulse function and the properties of the transform, one can easily generate very useful transform pairs for power signals, with a minimum of "hand-waving" arguments. Chapter 5 concludes with a discussion of the discrete Fourier transform as a method of numerically calculating the Fourier transform by a digital computer. Throughout the chapter we emphasize the physical nature of the transform by the use of examples taken primarily from communication theory applications.

Chapter 6 continues the discussion of transform methods for continuous-time signals by considering the Laplace transform. The Laplace transform is presented as a generalization of the Fourier transform. Both the bilateral and one-sided transforms are covered. In order to emphasize the unity inherent in transform methods, several examples are worked using both the Fourier and the Laplace transforms. These examples serve to point out that in most cases of a practical nature one does not need the Laplace transform, although numerical calculations with the Laplace transform are in some cases less complex. The Z-transform and the Fourier and Laplace transforms are integrated through a discussion of sampled time functions.

Our main purpose in including Chapter 7 is to further illustrate the relationships between continuous-time and discrete-time systems. In particular, we treat the problem of obtaining a desired continuous-time transfer function through the use of a discrete-time system (a digital filter). This chapter probably cannot be included in a one-semester course unless some earlier material is omitted. However, after the material treating the equivalent transfer function has been covered, students should be able to read through the remaining sections without undue difficulty. Alternatively, this chapter could be covered in a following semester or in a laboratory section.

Most new books have a way of building on the works of previous authors. In our case we are indebted to those authors mentioned in the references.

During the preparation of this book we have had many discussions with colleagues and graduate students. In particular, Min-Yen Wu and Jack Koplowitz offered many comments on the manuscript. Comments by Henry Hermes clarified many of our discussions in this book. The publisher's reviewers, John Thomas and Mac VanValkenburg, were most helpful. Teaching this material over the past several years has been an exciting and

rewarding experience, due in part to the interest of our students. Their suggestions and comments have improved the presentation markedly.

The typing was accomplished through the efforts of Mrs. Marie Krenz and Miss Judy Price. Their efforts are greatly appreciated. Finally, we would like to thank any kind readers who forward to us corrections and suggestions for improvements in this book.

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PART ONE

Time-Domain Techniques

1

Linear Systems

1.1 INTRODUCTION

The study of linear systems has been an essential part of formal undergraduate training for many years. Linear system analysis is important primarily because of its utility, for even though physical systems are never completely linear, a linear model is often appropriate over certain ranges of application. Also, a large body of mathematical theory is available for engineers and scientists to use in the analysis of such systems. In contrast, the analysis of nonlinear systems is essentially *ad hoc:* that is, each nonlinear system must be studied as one of a kind. There are no general methods of analysis and no general solutions.

The analysis of a given linear system is often facilitated by use of a particular class of input signals or a particular signal representation. Thus it is natural to include a study of signals and their properties in a study of linear systems. In later chapters we shall find this study especially fruitful.

As engineers, we are interested not only in the analysis but also in the synthesis of systems. In fact, it is the synthesis or design of systems that is the really creative portion of engineering. Yet, as in so many creative efforts, one must learn first how to analyze a system before one can proceed with system design. This work is directed primarily toward the analysis of certain classes of linear systems, although, because design and analysis are so intimately connected, this material will provide a basis for simple design.

We can divide the analysis of systems into three aspects:

(1) The development of a suitable mathematical model for the physical problem of interest. This portion of the analysis is concerned with obtaining the "equations of motion," boundary or initial conditions, parameter values, etc. This is the process wherein judgment, experience,

and experiments are combined to develop a suitable model. In some sense this first step is the hardest to develop formally.

(2) After a suitable model is obtained, one then solves the resultant equations to obtain solutions in various forms.

(3) One then relates or interprets the solution to the mathematical model in terms of the physical problem. It is to be hoped that the development in (1) has been accurate enough so that meaningful interpretations and predictions concerning the physical system can be made.

The primary emphasis of this work is on the second and third aspects mentioned above. The first step is essential but is probably better and more completely accomplished within a particular discipline. Thus, chemical engineers will learn to write equations of motion for chemical processes, electrical engineers for electrical circuits, and so on. After a model is obtained, one can consider various techniques for its solution and provide a basis for its mathematical interpretation.

Because linear models are so often used in all disciplines of engineering and science, this material is very useful. Perhaps the best way to point this fact out is to present examples from various physical problems. The only drawback to this method is that the reader may not always possess the necessary background to perform the first step in the analysis, to write the equations of motion. This problem is to be expected. As one gains familiarity with a given discipline, this first step becomes natural. Thus, without further apologies, we shall attempt to give physical examples from many fields without always attempting to develop a complete basis for the derivation of the equations of motion for a given system.

This material is presented as a summarizing work which brings together techniques and concepts that can be used to analyze a great variety of physical phenomena. This unity that one obtains is most useful and satisfying.

1.2 DEFINITIONS

We are primarily concerned with linear systems apart from their inherent physical structure. Thus, we often shall represent a linear system schematically as a box with inputs $x_1(t), x_2(t), \ldots, x_n(t)$ and outputs $y_1(t), y_2(t), \ldots, y_m(t)$ as in Figure 1.1. The inputs $x_i(t)$, $i = 1, 2, \ldots, n$ and

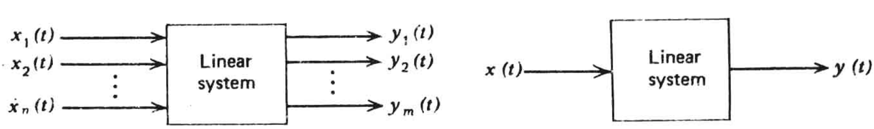

FIGURE 1.1
Schematic representations of a linear system.

outputs $y_j(t)$, $j = 1, 2, \ldots, m$ will, in general, be time signals: i.e., any physical variables of interest which vary with time. For the moment, let us focus on a single input, single output linear system. We shall consider multiple input-output systems in detail in Chapter 3.

Definition of a Linear System

The word linear suggests something pertaining to a straight line relationship. Thus, we might suspect that a linear system is one in which the output is in some sense proportional to the input. That is, if $x(t)$ gives rise to $y(t)$, then $\alpha x(t)$ gives rise to $\alpha y(t)$ for any constant α. In symbols, if

$$x(t) \rightarrow y(t)$$

then

$$\alpha x(t) \rightarrow \alpha y(t) \tag{1.1}$$

This property, called *homogeneity*, is a property of all linear systems. However, a linear system involves much more than (1.1). It must also possess the property of *superposition*. That is, if

$$x_1(t) \rightarrow y_1(t)$$

and

$$x_2(t) \rightarrow y_2(t)$$

then

$$x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t) \tag{1.2}$$

for some class of inputs $\{x(t)\}$. A system is linear if and only if superposition and homogeneity hold. We can combine (1.1) and (1.2) into a single equation. We define a system to be *linear* if and only if

$$\alpha x_1(t) + \beta x_2(t) \rightarrow \alpha y_1(t) + \beta y_2(t) \tag{1.3}$$

where α and β are constants. A convenient notation for the arrows of (1.1)–(1.3) is to use functional notation and represent the transformation of inputs into outputs by

$$y(t) = H[x(t)] \tag{1.4}$$

The system represented by (1.4) is linear if and only if H is a linear transformation: i.e., $H[\alpha x_1(t) + \beta x_2(t)] = \alpha H[x_1(t)] + \beta H[x_2(t)]$ Although the property of homogeneity can be inferred from the superposition property if α is a rational number, there are mathematical transformations which satisfy superposition and not homogeneity. However, these are really pathological examples, and they would not arise physically. Thus, we shall be content to verify the linearity of an input-output relation by checking superposition only.

EXAMPLE 1.1. Suppose a system has an input-output relation given by the linear equation

$$y(t) = ax(t) + b \tag{1.5}$$

Graphically, $x(t)$ and $y(t)$ are related as shown in Figure 1.2. Does Figure 1.2 represent a linear system? Consider the superposition property. If we apply an input $x_1(t)$, the corresponding output is $y_1(t) = ax_1(t) + b$. Similarly the input $x_2(t)$ gives as an output $y_2(t) = ax_2(t) + b$. If we now apply the input $x_1(t) + x_2(t)$, we obtain as an output $a(x_1(t) + x_2(t)) + b$ which is not equal to $(y_1(t) + y_2(t)) = a(x_1(t) + x_2(t)) + 2b$ unless $b = 0$. Thus, the system of Figure 1.2 is not linear even though the equation that relates $x(t)$ and $y(t)$ is a linear equation. It seems rather discouraging to have a system described by a linear equation and yet not be able to use linear analysis to analyze the system. We shall show in Section 1.5 how one can deal with this problem so that linear analysis can be applied to this system.

EXAMPLE 1.2. Consider the circuit shown in Figure 1.3. In this system, suppose that $x(t)$ is an input voltage and that $y(t)$ is an output voltage. As long as point A is less than 3 V, then

$$y(t) = x(t)/2 \tag{1.6}$$

Thus, $x_1(t)$ gives rise to $y_1(t) = x_1(t)/2$ and $x_2(t)$ gives rise to $y_2(t) = x_2(t)/2$. This means that $(x_1(t) + x_2(t))$ gives rise to $y_3(t) = (x_1(t) + x_2(t))/2$, which is $y_1(t) + y_2(t)$ (provided that $(x_1(t) + x_2(t)) < 3$). Thus, the system is linear as long as the diode is not conducting. This fact implies that point A must be less than 3 V: i.e., the system is linear for the class of signals $\{x(t)\}$, where $x(t)$ or any combination of $x(t)$'s is

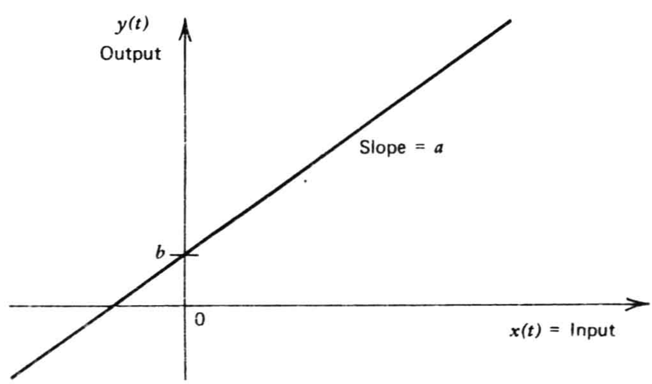

FIGURE 1.2