

CLASSICS IN MATHEMATICS

Herbert Federer

Geometric Measure Theory

几何测度论

Springer-Verlag

世界图书出版公司

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Geometric Measure Theory

Reprint of the 1969 Edition

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Originally published as Vol. 153 of the
Grundlehren der mathematischen Wissenschaften

Cataloging-in-Publication Data applied for

Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Federer, Herbert:
Geometric measure theory / Herbert Federer. - Reprint of the
1969 ed. - Berlin ; Heidelberg ; New York ; Barcelona ;
Budapest ; Hong Kong ; London ; Milan ; Paris ; Santa Clara ;
Singapore ; Tokyo : Springer, 1996
(Grundlehren der mathematischen Wissenschaften ; Vol. 153) (Classics
in mathematics)
ISBN 3-540-60656-4
NE: 1. GT

Mathematics Subject Classification (1991): 53C65, 46AXX

ISBN 3-540-60656-4 Springer-Verlag Berlin Heidelberg New York

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Springer-Verlag Berlin Heidelberg New York
a member of BertelsmannSpringer Science+Business Media GmbH

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Herbert Federer Geometric Measure Theory

Springer

Berlin

Heidelberg

New York

Barcelona

Budapest

Hong Kong

London

Milan

Paris

Santa Clara

Singapore

Tokyo

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Springer-Verlag Berlin · Heidelberg · New York 1969

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Geschäftsführende Herausgeber:

Prof. Dr. B. Eckmann

Eidgenössische Technische Hochschule Zürich

Prof. Dr. B. L. van der Waerden

Mathematisches Institut der Universität Zürich

Preface

During the last three decades the subject of geometric measure theory has developed from a collection of isolated special results into a cohesive body of basic knowledge with an ample natural structure of its own, and with strong ties to many other parts of mathematics. These advances have given us deeper perception of the analytic and topological foundations of geometry, and have provided new direction to the calculus of variations. Recently the methods of geometric measure theory have led to very substantial progress in the study of quite general elliptic variational problems, including the multidimensional problem of least area.

This book aims to fill the need for a comprehensive treatise on geometric measure theory. It contains a detailed exposition leading from the foundations of the theory to the most recent discoveries, including many results not previously published. It is intended both as a reference book for mature mathematicians and as a textbook for able students. The material of Chapter 2 can be covered in a first year graduate course on real analysis. Study of the later chapters is suitable preparation for research. Some knowledge of elementary set theory, topology, linear algebra and commutative ring theory is prerequisite for reading this book, but the treatment is selfcontained with regard to all those topics in multilinear algebra, analysis, differential geometry and algebraic topology which occur.

The formal presentation of the theory in Chapters 1 to 5 is preceded by a brief sketch of the main theme in the Introduction, which contains also some broad historical comments.

A systematic attempt has been made to identify, at the beginning of each chapter, the original sources of all relatively new and important material presented in the text. References to literature on certain additional topics, which this book does not treat in detail, appear in the body of the text. Some further related publications are listed only in the bibliography. All references to the bibliography are abbreviated in square brackets; for example [C 1] means the first listed work by C. Carathéodory.

The index is supplemented by a list of basic notations defined in the text, and a glossary of some standard notations which are used but not defined in the text.

I wish to thank Brown University and the National Science Foundation for supporting my work on this book, and to express appreciation of the efforts of my fellow mathematicians who helped with the project. Frederick J. Almgren Jr. and I had many stimulating discussions, in particular about his ideas presented in Section 5.3. Casper Goffman posed some interesting questions which inspired part of 4.5.9. Katsumi Nomizu showed me an elegant treatment of 5.4.13. William K. Allard read the whole manuscript with great care and contributed significantly, by many valuable queries and comments, to the accuracy of the final version. John E. Brothers, Lawrence R. Ernst, Josef Král, Arthur Sard and William P. Ziemer read parts of the manuscript and supplied very useful lists of errata.

The editors and personnel of Springer-Verlag have been unfailingly cooperative in all phases of publication of this book. I am grateful, in particular, to David Mumford for inviting me to contribute my work to the Grundlehren series, and to Klaus Peters for planning all the necessary arrangements with utmost consideration.

Herbert Federer

Providence, Rhode Island
January 1969

Druck: Strauss Offsetdruck, Mörlenbach
Verarbeitung: Schöffler, Grünstadt

书 名: Geometric Measure Theory
作 者: Herbert Federer
中译名: 几何测度论
出 版 者: 世界图书出版公司北京公司
印 刷 者: 北京世图印刷厂
发 行: 世界图书出版公司北京公司 (北京朝内大街 137 号 100010)
联系电话: 010-64015659, 64038347
电子信箱: kjsk@vip.sina.com
开 本: 24 印 张: 29
出版年代: 2004 年 11 月
书 号: 7-5062-6626-1/O · 479
版权登记: 图字:01-2004-5047
定 价: 88.00 元

世界图书出版公司北京公司已获得 Springer-Verlag 授权在中国大陆
独家重印发行。

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