GRADUATE STUDENT SERIES IN PHYSICS

INTRODUCTION TO GAUGE FIELD THEORY

DAVID BAILIN

ALEXANDER LOVE

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PREFACE

In the course of the 1970s important developments in the substance and form of particle physics have gradually rendered the excellent field theory texts of the 1950s and 1960s inadequate to the needs of postgraduate students. The main development in the substance of particle physics has been the emergence of gauge field theory as the basic framework for theories of the weak, electromagnetic and strong interactions. The main development on the formal side has been the increasing use of path (or functional) integral methods in the manipulation of quantum field theory, and the emphasis on the generating functionals for Green functions as basic objects in the theory. This latter development has gone hand-in-hand with the former because the comparative complexity and subtlety of non-Abelian gauge field theory has put efficient methods of proof and calculation at a premium.

It has been our objective in this book to introduce gauge field theory to the postgraduate student of theoretical particle physics entirely from a path integral standpoint without any reliance on the more traditional method of canonical quantisation. We have assumed that the reader already has a knowledge of relativistic quantum mechanics, but we have not assumed any prior knowledge of quantum field theory. We believe that it is possible for the postgraduate student to make his first encounter with scalar field theory in the path integral formalism, and to proceed from there to gauge field theory. No attempt at mathematical rigour has been made, though we have found it appropriate to indicate how well-defined path integrals may be obtained by an analytic continuation to Euclidean space.

We have chosen for the contents of this book those topics which we believe form a foundation for a knowledge of modern relativistic quantum field theory. Some topics inevitably had to be included, such as the path integral approach to scalar field theory, path integrals over Grassmann variables necessary for fermion field theories, the Faddeev–Popov quantisation procedure for non-Abelian gauge field theory, spontaneous breaking of symmetry in gauge theories, and the renormalisation group equation and asymptotic freedom. At a more concrete level this enables us to discuss quantum chromodynamics (QCD) and electroweak theory. Some topics have been included as foundation material which might not have appeared if the book had been written at a slightly earlier date. For example, we have inserted a chapter on field theory at non-zero temperature, in view of the large body of literature that now exists on the application of gauge field theory to cosmology. We have also included a chapter on grand unified theory. Some topics we have omitted from this introductory text, such as an extensive discussion of the results of perturbative QCD (though some applications have been discussed in the text), non-perturbative QCD, and supersymmetry.

We owe much to Professor R G Moorhouse who suggested that we should write this book, and to many colleagues, including P Frampton, A Sirlin, J Cole, T Muta, H F Jones, D R T Jones, D Lancaster, J Fleischer, Z Hioki and G Barton, for the physics they have taught us. We are very grateful to Mrs S Pearson and Ms A Clark for their very careful and speedy typing of the manuscript. Finally, we are greatly indebted to our wives, to whom this book is dedicated, for their invaluable encouragement throughout the writing of this book.

David Bailin
Alexander Love

CONTENTS

1

PATH INTEGRALS

There are two widely used approaches to quantum field theory. The first is based on field operators and the canonical quantisation of these operator fields, and will not be discussed in this book. The second approach, as we shall see in Chapter 4, involves path integrals[1] over classical fields, and it is upon this latter approach that this book relies for its derivations. In this chapter, the idea of path integrals (or functional integrals) will be developed in a very intuitive way without any attempt at mathematical precision or rigour. Instead, the analogy between vectors and functions, and between matrices and differential operators on functions will be exploited extensively. Since very few path integrals can be performed exactly, we shall concentrate on Gaussian path integrals[2]. These are important in their own right, but much more so because they can be used in approximation schemes when the exact path integral is intractable, as we shall see in later chapters.

Our starting point is the ordinary Gaussian integral

$$\int_{-\infty}^{\infty} \mathrm{d}y \exp(-\tfrac{1}{2}ay^2) = (2\pi)^{1/2}a^{-1/2} \qquad a>0. \tag{1.1}$$

This may be generalised to the integral over n real variables

$$\int_{-\infty}^{\infty} \mathrm{d}y_1 \ldots \mathrm{d}y_n \exp(-\tfrac{1}{2}Y^{\mathrm{T}}\mathbf{A}Y) = (2\pi)^{n/2}(\det \mathbf{A})^{-1/2} \tag{1.2}$$

where $\mathbf{A}$ is a real symmetric positive definite matrix, Y is the column vector with components $(y_1 \ldots, y_n)$, the transpose of Y is denoted by Y^{T}, and each integral is understood to be over the range $(-\infty, \infty)$. Equation (1.2) is easily derived by diagonalising $\mathbf{A}$, when the n-dimensional integration becomes a product of n integrals of the form (1.1). It will prove convenient to write

$$\det \mathbf{A} = \exp \ln \det \mathbf{A} = \exp \mathrm{Tr} \ln \mathbf{A} \tag{1.3}$$

where Tr denotes the trace of the matrix. The identity

$$\ln \det \mathbf{A} = \mathrm{Tr} \ln \mathbf{A} \tag{1.4}$$

is most easily proved by diagonalising $\mathbf{A}$. Equation (1.2) can now be written as

$$(2\pi)^{-n/2} \int_{-\infty}^{\infty} \mathrm{d}y_1 \ldots \mathrm{d}y_n \exp(-\tfrac{1}{2}Y^{\mathrm{T}}\mathbf{A}Y) = \exp(-\tfrac{1}{2}\mathrm{Tr} \ln \mathbf{A}). \tag{1.5}$$

We wish to generalise (1.5) to the case where the integration is over the continuous infinity of components of a function $\varphi(x)$ rather than over the finite number of components of the column vector Y. Such an integral is called a path (or functional) integral. Proceeding intuitively[2] we write

$$\int \mathscr{D}\varphi \exp\left(-\tfrac{1}{2}\int dx' \int dx\varphi(x')A(x', x)\varphi(x)\right) = \exp(-\tfrac{1}{2}\,\mathrm{Tr}\ln \mathbf{A}) \qquad (1.6)$$

where we use the symbol $\mathscr{D}$ for path integration, and we assume that the integral has been defined in such a way as to remove any normalisation factor (corresponding to the factor $(2\pi)^{-n/2}$ in (1.5)). The integrals over x' and x are assumed to be one-dimensional integrals over the range $(-\infty, \infty)$. However, the treatment generalises trivially to the case where dx is replaced by d^4x, and the integration is over the whole four-dimensional space. The trace in (1.6) may be evaluated by Fourier transforming. For example, consider the case

$$A(x', x) = \left(\frac{\partial}{\partial x'}\frac{\partial}{\partial x} + r\right)\delta(x' - x) \qquad (1.7)$$

where r is a constant. (This is closely related to situations we shall encounter in later chapters.) The one-dimensional Dirac delta function has the integral representation

$$\delta(x' - x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x'-x)}. \qquad (1.8)$$

Thus

$$A(x', x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip(x'-x)}(p^2 + r) \qquad (1.9)$$

and

$$\mathrm{Tr}\ln \mathbf{A} = \int dx \int \frac{dp}{2\pi} \ln(p^2 + r) \qquad (1.10)$$

where to take the trace we have set $x' = x$ and integrated over all values of x, since we have a continuous infinity of degrees of freedom.

A slight generalisation can be made by introducing a linear term in (1.5).

$$(2\pi)^{-n/2}\int_{-\infty}^{\infty} dy_1 \ldots dy_n \exp(-\tfrac{1}{2}Y^{\mathrm{T}}\mathbf{A}Y + \rho^{\mathrm{T}}Y)$$
$$= \exp(-\tfrac{1}{2}\mathrm{Tr}\ln \mathbf{A})\exp(\tfrac{1}{2}\rho^{\mathrm{T}}\mathbf{A}^{-1}\rho) \qquad (1.11)$$

where ρ is a given column vector, and $\mathbf{A}^{-1}$ exists because $\mathbf{A}$ is positive-definite. Equation (1.11) is derived from (1.5) by completing the square,

$$Y^{\mathrm{T}}\mathbf{A}Y - 2\rho^{\mathrm{T}}Y = (Y - \mathbf{A}^{-1}\rho)^{\mathrm{T}}\mathbf{A}(Y - \mathbf{A}^{-1}\rho) - \rho^{\mathrm{T}}\mathbf{A}^{-1}\rho \qquad (1.12)$$

and making the change of variable

$$Y' = Y - \mathbf{A}^{-1}\rho. \qquad (1.13)$$

The corresponding path integral is

$$\int \mathscr{D}\varphi \exp\left(-\tfrac{1}{2}\int dx' \int dx\varphi(x')A(x',x)\varphi(x) + \int dx\rho(x)\varphi(x)\right)$$

$$= \exp(-\tfrac{1}{2}\,\mathrm{Tr}\ln \mathbf{A})\exp\left(\tfrac{1}{2}\int dx' \int dx\rho(x')A^{-1}(x',x)\rho(x)\right). \quad (1.14)$$

where $\rho(x)$ is a given function. In (1.14), $A^{-1}(x', x)$ is easily evaluated from the Fourier transform of $A(x', x)$. Thus, with $A(x', x)$ as in (1.7), we have from (1.9),

$$A^{-1}(x',x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, e^{ip(x'-x)}(p^2+r)^{-1}. \quad (1.15)$$

Equations (1.11) and (1.14) enable us to carry out somewhat more general integrals than Gaussian integrals. If we differentiate with respect to $\rho_{m_1}, \rho_{m_2}, \ldots, \rho_{m_p}$ at $\rho=0$ in (1.11) we obtain

$$(2\pi)^{-n/2}\int_{-\infty}^{\infty} dy_1 \ldots dy_n\, y_{m_1} \ldots y_{m_p} \exp(-\tfrac{1}{2}Y^{\mathrm{T}}\mathbf{A}Y)$$

$$= \exp(-\tfrac{1}{2}\,\mathrm{Tr}\ln \mathbf{A})(A^{-1}_{m_1m_2} \ldots A^{-1}_{m_{p-1}m_p} + \text{permutations}) \quad (1.16)$$

when p is even, and zero when p is odd.

Generalised to the path integral case

$$\int \mathscr{D}\varphi\, \varphi(x_1) \ldots \varphi(x_p) \exp\left(-\tfrac{1}{2}\int dx' \int dx\, \varphi(x')A(x',x)\varphi(x)\right)$$

$$= \exp(-\tfrac{1}{2}\,\mathrm{Tr}\ln \mathbf{A})(A^{-1}(x_1,x_2) \ldots A^{-1}(x_{p-1},x_p) + \text{permutations}). \quad (1.17)$$

To carry through the differentiations in the path integral case we understand the derivatives to be functional derivatives $\delta/\delta\rho(x_1), \ldots, \delta/\delta\rho(x_p)$, where by definition

$$\frac{\delta}{\delta\rho(x_i)}\left(\int dx\, \rho(x)\varphi(x)\right) = \varphi(x_i) \qquad i=1,\ldots,p. \quad (1.18)$$

We have been discussing a real column vector or a real function $\varphi(x)$. The discussion is easily extended to the case of complex column vectors or functions. Thus, for example,

$$(2\pi)^{-n}\int dz_1\, dz_1^* \ldots dz_n\, dz_n^* \exp(-Z^\dagger \mathbf{A} Z) = (\det \mathbf{A})^{-1} = \exp(-\mathrm{Tr}\ln \mathbf{A}) \quad (1.19)$$

where $\mathbf{A}$ is a Hermitian matrix, Z is the complex column vector with components $(z_1, \ldots, z_n)$, $Z^\dagger = (Z^*)^{\mathrm{T}}$, and

$$\int dz\, dz^* \equiv 2\int d(\mathrm{Re}\, z)\, d(\mathrm{Im}\, z). \quad (1.20)$$

The corresponding path integral is

$$\int \mathscr{D}\varphi \mathscr{D}\varphi^* \exp\left(-\int dx' \int dx\, \varphi^*(x')A(x',x)\varphi(x)\right) = \exp(-\operatorname{Tr}\ln \mathbf{A}). \tag{1.21}$$

So far we have been assuming that the reader is making the intuitive leap from a column vector with a finite number of components to a function with a continuous infinity of components. We can put path integrals on a (slightly) more formal basis, as follows[1]. Suppose that the x and x' integrations in (1.6) are over the finite range from X to $\bar{X}$. We can take the limit of an infinite range of integration at the end of our discussion. Divide the range up into $N+1$ equal segments of length ε

$$(N+1)\varepsilon = \bar{X} - X. \tag{1.22}$$

Let the steps begin at $x_0 = x, x_1, x_2, \ldots, x_N$, and adopt the notations

$$\varphi_i = \varphi(x_i) \qquad A_{jk} = A(x_j, x_k). \tag{1.23}$$

Then we may define the Gaussian path integral as follows:

$$\begin{aligned}\int \mathscr{D}\varphi \exp&\left(-\tfrac{1}{2}\int dx' \int dx\, \varphi(x')A(x',x)\,\varphi(x)\right)\\ &= \lim_{N\to\infty} (2\pi)^{-N/2} \prod_{i=1}^{N} \int d\varphi_i \exp\left(-\tfrac{1}{2}\sum_{j,k} \varphi_j A_{jk}\varphi_k\right)\\ &= \lim_{N\to\infty} (2\pi)^{-N/2} \prod_{i=1}^{N} \int d\varphi_i \exp(-\tfrac{1}{2}\boldsymbol{\varphi}^{\mathrm{T}}\mathbf{A}\boldsymbol{\varphi})\end{aligned} \tag{1.24}$$

where $\boldsymbol{\varphi}$ is the column vector with components $(\varphi_1, \ldots, \varphi_N)$, and $\mathbf{A}$ is the matrix with entries A_{jk}. In the case where we allow the range of integration $(X, \bar{X})$ to become the interval $(-\infty, \infty)$, we may perform the Gaussian integral to obtain the result of equation (1.6). We must, of course, interpret $\lim_{N\to\infty} \exp(-\frac{1}{2}\operatorname{Tr}\ln \mathbf{A})$, where $\mathbf{A}$ is the matrix, as $\exp(-\frac{1}{2}\operatorname{Tr}\ln A)$, where A is $A(x', x)$.

Problem

1.1 Derive (1.19) from (1.5).

References

1 Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)

2 We follow most closely the approach of
Coleman S 1973 *Lectures given at the 1973 International Summer School of Physics, Ettore Majorana.*

2

PATH INTEGRALS IN NON-RELATIVISTIC QUANTUM MECHANICS

2.1 Transition amplitudes as path integrals

It was shown by Feynman (following a lead by Dirac) that quantum mechanics could be formulated in terms of path integrals[1,2]. We shall discuss this approach to quantum mechanics in some detail since it provides the key to the path integral formulation of quantum field theory. For simplicity, we shall consider in the first instance a system described by a single generalised coordinate Q, with a conjugate momentum P. When corresponding quantum mechanical operators for Q are required we shall use the notation $\hat{Q}_H$ in the Heisenberg picture, and $\hat{Q}_S$ in the Schödinger picture. We denote the eigenstates of $\hat{Q}_S$ by $|q\rangle_S$:

$$\hat{Q}_S|q\rangle_S = q|q\rangle_S. \tag{2.1}$$

Since $\hat{Q}_H$ is time-dependent, so are its eigenstates, which we denote by $|q, t\rangle$:

$$\hat{Q}_H(t)|q, t\rangle = q|q, t\rangle. \tag{2.2}$$

(It should, of course, be remembered that the physical state vectors, as opposed to the eigenstates of $\hat{Q}_H(t)$, are time-independent in the Heisenberg picture.) The relevant connections between the two pictures are

$$\hat{Q}_H(t) = e^{i\hat{H}t/\hbar}\,\hat{Q}_S\,e^{-i\hat{H}t/\hbar} \tag{2.3}$$

and

$$|q, t\rangle = e^{i\hat{H}t/\hbar}\,|q\rangle_S \tag{2.4}$$

where $\hat{H}$ is the (time-independent) Hamiltonian operator.

The probability amplitude that a system which was in the eigenstate $|q', t'\rangle$ at time t' will be found to have the value q'' of Q at time t'' is

$$\langle q'', t''|q', t'\rangle = {}_S\langle q''|e^{-i\hat{H}(t''-t')/\hbar}|q'\rangle_S. \tag{2.5}$$

This transition amplitude may be expressed as a path integral by dividing the time interval from t' to t'' into $N+1$ small steps of equal length ε, with

$$(N+1)\varepsilon = t'' - t'. \tag{2.6}$$

Let the steps begin at $t', t_1, t_2, \ldots, t_N$. The eigenstates of $\hat{Q}_H(t)$ form a complete set for any given value of t. Thus

$$\langle q'', t''|q', t'\rangle = \prod_{j=1}^{N} \int \mathrm{d}q_j \langle q'', t''|q_N, t_N\rangle\langle q_N, t_N|q_{N-1}, t_{N-1}\rangle \ldots \langle q_1, t_1|q', t'\rangle. \tag{2.7}$$

We need to study $\langle q_{j+1}, t_{j+1}|q_j, t_j\rangle$ as N becomes very large and the step length ε becomes very small. The discussion is much simplified if the Hamiltonian is of the form

$$H(Q, P) = \frac{P^2}{2m} + V(Q). \tag{2.8}$$

(Simplification results because products of P with Q are not involved, and problems of order associated with the lack of commutativity of the corresponding operators are alleviated.) Applying equation (2.5) to first non-trivial order in ε,

$$\langle q_{j+1}, t_{j+1}|q_j, t_j\rangle \approx {}_{\mathrm{S}}\left\langle q_{j+1}\left|1 - \frac{\mathrm{i}\hat{H}\varepsilon}{\hbar}\right|q_j\right\rangle_{\mathrm{S}}. \tag{2.9}$$

But

$${}_{\mathrm{S}}\langle q_{j+1}|\hat{H}|q_j\rangle_{\mathrm{S}} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q_j^2}\right){}_{\mathrm{S}}\langle q_{j+1}|q_j\rangle_{\mathrm{S}} \tag{2.10}$$

and using the usual integral representation of the Dirac δ function

$${}_{\mathrm{S}}\langle q_{j+1}|q_j\rangle_{\mathrm{S}} = \delta(q_{j+1} - q_j) = \hbar^{-1}\int_{-\infty}^{\infty}\frac{\mathrm{d}p_j}{2\pi}\exp[\mathrm{i}p_j(q_{j+1} - q_j)\hbar^{-1}] \tag{2.11}$$

where we have chosen to write the integration variable as $p_j/\hbar$. Using (2.11) in (2.10) gives

$${}_{\mathrm{S}}\langle q_{j+1}|\hat{H}|q_j\rangle_{\mathrm{S}} = \hbar^{-1}\int\frac{\mathrm{d}p_j}{2\pi}\left(\frac{\hbar^2 p_j^2}{2m} + V(q_j)\right)\exp[\mathrm{i}p_j(q_{j+1} - q_j)\hbar^{-1}] \tag{2.12}$$

$$= \hbar^{-1}\int\frac{\mathrm{d}p_j}{2\pi}\exp[\mathrm{i}p_j(q_{j+1} - q_j)\hbar^{-1}]H(q_j, p_j). \tag{2.13}$$

We may now rewrite (2.9) as

$$\langle q_{j+1}, t_{j+1}|q_j, t_j\rangle \approx \hbar^{-1}\int\frac{\mathrm{d}p_j}{2\pi}\exp[\mathrm{i}p_j(q_{j+1} - q_j)\hbar^{-1}](1 - \mathrm{i}\varepsilon\hbar^{-1}H(q_j, p_j)). \tag{2.14}$$

Still working to first non-trivial order in ε, we write the integrand in (2.14) as an exponential

$$\langle q_{j+1}, t_{j+1}|q_j, t_j\rangle \approx \hbar^{-1}\int \frac{dp_j}{2\pi} \exp\{i\hbar^{-1}\varepsilon[p_j(q_{j+1}-q_j)\varepsilon^{-1} - H(q_j, p_j)]\}. \quad (2.15)$$

Returning to (2.7), the transition amplitude may now be expressed in the form

$$\langle q'', t''|q', t'\rangle \approx \prod_{j=1}^{N}\int dq_j \prod_{j=0}^{N}\frac{dp_j}{2\pi\hbar}\exp\left(i\hbar^{-1}\varepsilon\sum_{j=0}^{N}[p_j(q_{j+1}-q_j)\varepsilon^{-1} - H(q_j, p_j)]\right) \quad (2.16)$$

where we have written

$$q_0 = q' \qquad q_{N+1} = q''. \quad (2.17)$$

Taking the limit $N \to \infty$ with $(N+1)\varepsilon$ fixed as in (2.6), we obtain the transition amplitude as a path integral

$$\langle q'', t''|q', t'\rangle \propto \int \mathscr{D}q \int \mathscr{D}p \exp i\hbar^{-1}\int_{t'}^{t''} dt(p\dot{q} - H(p, q)) \quad (2.18)$$

where the integration is over all functions $p(t)$, and over all functions $q(t)$ which obey the boundary conditions

$$q(t') = q' \qquad q(t'') = q''. \quad (2.19)$$

The result is more general than the case to which we have restricted ourselves in (2.8).

When the Hamiltonian is given by (2.8), the p_j integrations in (2.15) and (2.16) may be carried out (formally). We complete the square by making the change of variables

$$\tilde{p}_j = p_j - m\varepsilon^{-1}(q_{j+1} - q_j) \quad (2.20)$$

and perform the integrations formally by pretending that $i\varepsilon$ is real (continuation to imaginary time). We then have Gaussian integrals and obtain

$$\langle q_{j+1}, t_{j+1}|q_j, t_j\rangle \approx (2\pi i\varepsilon\hbar/m)^{-1/2}\exp\{i\hbar^{-1}\varepsilon[\tfrac{1}{2}m\varepsilon^{-2}(q_{j+1}-q_j)^2 - V(q_j)]\}. \quad (2.21)$$

Using (2.21) in (2.7) gives

$$\langle q'', t''|q', t'\rangle \approx (2\pi i\varepsilon\hbar/m)^{-(N+1)/2}\prod_{j=1}^{N}\int dq_j \exp\left(i\hbar^{-1}\varepsilon\sum_{j=0}^{N}[\tfrac{1}{2}m\varepsilon^{-2}(q_{j+1}-q_j)^2 - V(q_j)]\right). \quad (2.22)$$

Taking the limit $N \to \infty$ with $(N+1)\varepsilon$ fixed as in (2.6) yields the path integral