Lecture Notes in Mathematics

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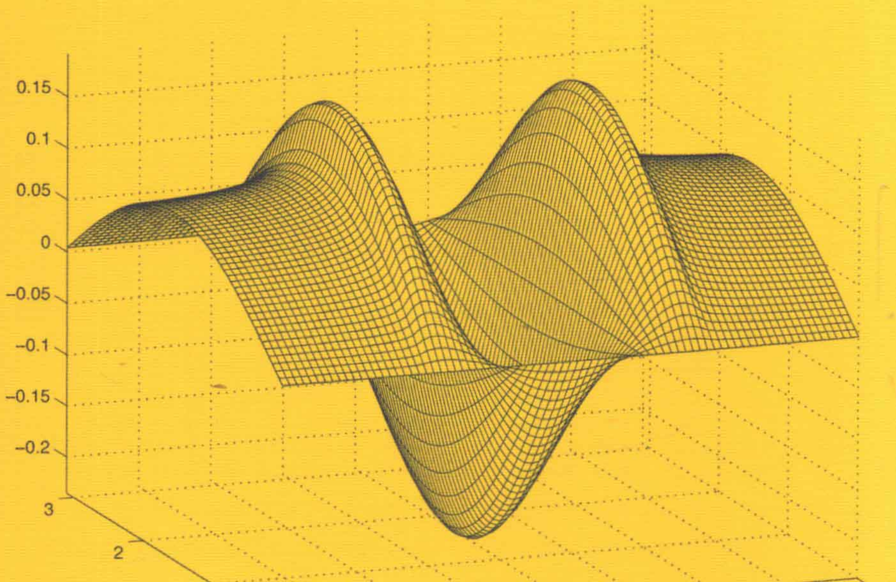
Thomas Schuster

The Method of Approximate Inverse: Theory and Applications



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The Method of Approximate Inverse: Theory and Applications

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Lecture Notes in Mathematics

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Lecture Notes in Mathematics 1906

Dedicated to Petra, for her patience, understanding, and love

Preface

Many questions and applications in natural science, engineering, industry or medical imaging lead to inverse problems, that is: given some measured data one tries to recover a searched for quantity. These problems are of growing interest in all these disciplines and thus there is a great need for modern and stable solvers for these problems. A prominent example of an inverse problem is the problem of computerized tomography: From measured X-ray attenuation coefficients one has to calculate densities in human tissue. Mathematically inverse problems often are described as operator equations of first kind

$$\mathbf{A}f = g\,, \tag{0.1}$$

where $\mathbf{A} : X \to Y$ is a bounded operator acting on appropriate topological spaces X and Y. In case of 2D computerized tomography the mapping $\mathbf{A}$ is given by the Radon transform. Typically these operators have unbounded inverses $\mathbf{A}^{-1}$, if they are invertible at all. For instance if $\mathbf{A}$ is compact with infinite dimensional range, then $\mathbf{A}^{-1}$ is not continuous. In case of Hilbert spaces X and Y the generalized inverse $\mathbf{A}^{\dagger}$ exists and has a dense domain. But $\mathbf{A}^{\dagger}$ is bounded if and only if the range of $\mathbf{A}$ is closed which is not satisfied for compact $\mathbf{A}$. In applications the exact data g is noise contaminated e.g. by the measurement process or discretization errors. Noisy data g^{ε} lead to an useless solution $f^{\varepsilon} = \mathbf{A}^{-1}g^{\varepsilon}$ or $f^{\varepsilon} = \mathbf{A}^{\dagger}g^{\varepsilon}$ in the sense that the error $f - f^{\varepsilon}$ is unacceptably large. Hence, the stable solution of equations like (0.1) with noisy right-hand side g^{ε} require regularization methods R_{γ}. The mappings R_{γ} are bounded operators which converge pointwise to the unbounded generalized inverse $\mathbf{A}^{\dagger}$. Many regularization techniques have been developed over the last decades such as the truncated singular value decomposition, the Tikhonov-Phillips regularization or iterative methods such as the Landweber method and the method of conjugate gradients (CG-method) to name only the most popular ones.

A powerful tool which subsumes a whole family of regularization techniques is the method of approximate inverse. This method uses the duality of the operator and the spaces where it acts on. It calculates approximations to the exact solution by smoothing it with mollifiers which are approximations to Dirac's delta distribution and attenuate high frequencies contained in the solution. The method consists then of the evaluation of the measured data with so called reconstruction kernels. The reconstruction kernels themselves are solutions of an equation involving the dual operator and the mollifier and can be precomputed before the measurement process starts. A further feature of the method is its flexibility: it can be adjusted to the operator and the underlying spaces to improve the efficiency. The first idea of solving linear operator equations by mollifier methods arose in 1990 by LOUIS AND MAASS [71] and it was Louis, who published its first fundamental properties [66] and showed its regularization property [68]. RIEDER AND SCHUSTER [101, 102] derived a setting of the method for operators between arbitrary Hilbert spaces and proved convergence with rates and stability. An extension of the method to spaces of distributions was done by SCHUSTER, QUINTO [115]. The article SCHÖPFER ET AL. [107] must be seen as a first step to realize this technique in Banach spaces.

This monograph contains a comprehensive outline of the theoretical aspects of the method of approximate inverse (Part I) as well as applications of the method to different inverse problems arising in medical imaging and nondestructive testing (Parts II-IV). Part I gives a brief introduction to inverse problems and regularization methods and introduces then the approximate inverse on spaces of square integrable functions, where the Radon transform serves as a first example. We then go one step further and present the abstract setup of solving semi-discrete operator equations between arbitrary Hilbert spaces by the method of approximate inverse. Semi-discrete operator equations are of wide interest since in practical applications only a finite number of measured data is available. Part I ends with an extension of the theory to spaces of distributions. Part II puts life into the theoretical considerations of Part I and demonstrates their transfer to the problem of 3D Doppler tomography. Doppler tomography belongs to the area of medical imaging and means the problem of recovering the velocity field of a moving fluid from ultrasonic Doppler measurements. It is outlined how the method of approximate inverse leads to a solver of filtered backprojection type on the one hand and can be involved in the construction of defect correction methods on the other hand. In SONAR (SOund in NAvigation and Radiation) and SAR (Synthetic Aperture Radar) the problem arises of inverting a spherical mean operator. If the center set consists of a hyperplane this operator can no longer be described as a bounded mapping between Hilbert or Banach spaces, but it extends to a linear, continuous mapping between spaces of tempered distributions. Part III of the book presents the extension of the method to distribution spaces and shows its performance when being applied to the spherical mean operator. Further applications such as X-ray diffractometry, which is a sort of non-destructive testing, thermoacoustic tomography, where the spherical mean operator is involved, too, but with spheres as center sets, and 3D computerized tomography are the contents of Part IV. The book contains plenty of numerical results which prove that the method is well suited to cope with inverse problems in practical situations and each part is completed by a conclusion and future perspectives.

This monograph is an extended version of my habilitation thesis which I submitted at the Saarland University Saarbrücken (Germany) in 2004. The mathematical results contained therein would have been impossible to accomplish without some important people accompanying my scientific way now for many years. Thus, the first person I would like to thank is my teacher Prof. Dr. A.K. Louis who introduced me to the area of approximate inverse many years ago and who supported me all the time. Part II of the book was the result of an intensive collaboration with Prof. Dr. A. Rieder between 2000 and 2004 and I am still thankful for conveying his rich experience in approximation theory to me. Part III of the book was the result of an one year stay at Tufts University in Medford (USA) at the chair of Prof. Dr. E.T. Quinto, an acclaimed expert in integral geometry and numerical mathematics. I owe him many useful pointers with respect to the extension of the approximate inverse to distribution spaces and I will never forget his hospitality. I am further indebted to Dr. R. Müller for a very careful review of the manuscript.

Thomas Schuster Hamburg, January 2007

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Part III Application to the spherical mean operator

Part IV Further Applications

Part I

Inverse and Semi-discrete Problems

Many applications in natural science, industry, medicine and engineering can be concisely described by an operator equation of first kind

$$\mathbf{A}f = g\,. \tag{0.2}$$

Here, g can be seen as a set of measurement data and f is the quantity we are searching for. The mapping $\mathbf{A}$ tells how f and the data g are connected to each other. The properties of $\mathbf{A}$ have influence on the mathematical solution of (0.2). A problem like (0.2), that is observe g and find the solution f, is called an *inverse problem*.

The method of approximate inverse is a powerful and versatile tool for solving inverse problems. Articles as e.g. LOUIS [67], LOUIS, ABDULLAH [1], JONAS, LOUIS [49], LOUIS, SCHUSTER [75], RIEDER, SCHUSTER [102, 103], SCHUSTER [110] and SCHUSTER, QUINTO [115] prove this. The idea is to compute a smoothing f_γ of f. If f is a function this can be done by convolving f with e_γ which is a smooth function such as the Gaussian kernel. Then, using the duality of $\mathbf{A}$ and $\mathbf{A}^*$, f_γ can be calculated by evaluating inner products of the measured data g with a so-called reconstruction kernel v_γ which is performed in an efficient and stable way. The approximate inverse represents a class of *regularization methods*, which by its flexibility allows an adjustment to the underlying problem at a high level. Further we will see how invariance properties of $\mathbf{A}$ enhance the efficacy of the resulting inversion scheme. In practical situations equation (0.2) often is not an appropriate setting. Instead of the 'complete' function g we maybe have only a finite number of observations, e.g. moments or point evaluations, of g at hand. Thus, a semi-discrete setting

$$\mathbf{A}_n f = g_n$$

is more suited to describe real-world problems. Here, $\mathbf{A}_n$ emerges from $\mathbf{A}$ and g_n from g by an application of the so-called *observation operator* which models the measurement procedure and contains all information about the measurement device, e.g. its geometry. In order to prove strong convergence we prefer to investigate the semi-discrete rather than a fully discrete setting, though the latter one seems to be more useful from a practical point of view. Given finitely many computed moments $\langle f, e_i\rangle$ of f we obtain an approximation to f using an appropriate interpolation mapping. All these issues are subject of the first part of the book.

Part I includes five chapters. The first chapter provides essential facts from the theory of regularization methods for inverse problems. The basic idea of the approximate inverse is then demonstrated for linear, bounded mappings between L^2-spaces in the second chapter along with the two-dimensional Radon transform as a first application. In Chapter 3 we extend the method to semi-discrete problems in arbitrary Hilbert spaces. We show that the approximate inverse in fact is a regularization method and present a rigorous convergence and stability analysis. Chapter 4 finally consists of the presentation of a framework for solving semi-discrete equations in distribution spaces, which has important applications in SAR and SONAR. We furthermore sketch the idea of an error analysis for this case, too. We finish this part with some remarks collected in Chapter 5.

1

Ill-posed problems and regularization methods

We start by presenting the essential concepts for regularizing ill-posed operator equations of first kind. We refer the reader who is interested in a comprehensive treatise of this subject to the standard textbooks of Louis [65], Engl, Hanke and Neubauer [26], Rieder [99], Tarantola [125] and Hofmann [46][1].

We only consider the case of linear and bounded operators $\mathbf{A}$ between Hilbert spaces X and Y. The set of linear and bounded operators between X and Y is denoted by $\mathcal{L}(X, Y)$. Our aim is to investigate the solution of

$$\mathbf{A}f = g\,, \tag{1.1}$$

where $g \in Y$ is a given set of data and $f \in X$ is the quantity we want to determine. In case that $\mathbf{A}$ has a bounded inverse $\mathbf{A}^{-1}$ we obtain f by simply calculating $f = \mathbf{A}^{-1}g$. In this situation we call (1.1) *well-posed*. Unfortunately in many real applications $\mathbf{A}$ is not invertible at all. Even if $\mathbf{A}^{-1}$ exists, then the inverse might not be bounded, e.g. when $\mathbf{A}$ is compact with infinite dimensional range. Moreover, in real world problems the exact data g might not be available, but only a noise contaminated set of measurements $g^\varepsilon \in Y$ with

$$\|g^\varepsilon - g\|_Y = \varepsilon\,.$$

The corresponding solution $f^\varepsilon = \mathbf{A}^{-1}g^\varepsilon$ then usually does not converge to f if $\varepsilon \to 0$ and the defect

$$\|f^\varepsilon - f\|_X$$

can be tremendously large. The solution of equations like (1.1) where $\mathbf{A}$ has no bounded inverse $\mathbf{A}^{-1}$ is called an *ill-posed problem* due to Hadamard.

Since often $\mathbf{A}^{-1}$ does not exist, the aim is to generalize what we understand by a *solution* of an equation like (1.1). To this end consider the defect

$$d(f) = \|\mathbf{A}f - g\|_Y\,,$$

[1] References [26], [125] are in English.