LINEAR ALGEBRA

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linear algebra

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To Our Families

preface

The language and concepts of matrix theory and, more generally, of linear algebra have come into widespread usage in the social and natural sciences. In addition, linear algebra continues to be of great importance in modern treatments of geometry and analysis.

The primary purpose of this book is to present a careful treatment of the principal topics of linear algebra and to illustrate the power of the subject through a variety of applications. Although the only formal prerequisite for the book is a one-year course in calculus, the material in Chapters 6 and 7 requires the mathematical sophistication of typical college juniors and seniors (who may or may not have had some previous exposure to the subject).

The book is organized to permit a number of different courses (ranging from three to six semester hours in length) to be taught from it. The core material (vector spaces, linear transformations and matrices, systems of linear equations, determinants, and diagonalization) is found in Chapters 1 through 5. The remaining chapters, treating canonical forms and inner product spaces, are completely independent and may be studied in any order. In addition, throughout the book are a variety of applications to such areas as differential equations, economics, geometry, and physics. These applications are not central to the mathematical development, however, and may be excluded at the discretion of the instructor.

We have attempted to make it possible for many of the important topics of linear algebra to be covered in a one-semester course. This goal has led us to develop the major topics with fewer unnecessary preliminaries than in a traditional approach. (Our treatment of the Jordan canonical form, for instance, does not require any theory of polynomials.) The resulting economy permits us to cover most of the book (omitting many of the optional sections and a detailed discussion of determinants) in a one-semester four-hour course for students who have had some prior exposure to linear algebra.

Chapter 1 of the book presents the basic theory of finite-dimensional vector spaces—subspaces, linear combinations, linear dependence and independence, bases, and dimension. The chapter concludes with an optional section in which we prove the existence of a basis in infinite-dimensional vector spaces.

Linear transformations and their relationship to matrices are the subject of Chapter 2. We discuss there the null space and range of a linear transformation, matrix representations of a transformation, isomorphisms, and change of coordinates. Optional sections on dual spaces and homogeneous linear differential equations end the chapter.

The applications of vector space theory and linear transformations to systems of linear equations are found in Chapter 3. We have chosen to defer this important subject so that it can be presented as a consequence of the preceding material. This approach allows the familiar topic of linear systems to illuminate the abstract theory and permits us to avoid messy matrix computations in the presentation of Chapters 1 and 2. There will be occasional examples in these chapters, however, where we shall want to solve systems of linear equations. (Of course, these examples will not be a part of the theoretical development.) The necessary background is contained in Section 1.4.

Determinants, the subject of Chapter 4, are of much less importance than they once were. In a short course we prefer to treat determinants lightly so that more time may be devoted to the material in Chapters 5 through 7. Consequently we have presented two alternatives in Chapter 4—a complete development of the theory (Sections 4.1 through 4.4) and a summary of the important facts that are needed for the remaining chapters (Section 4.5).

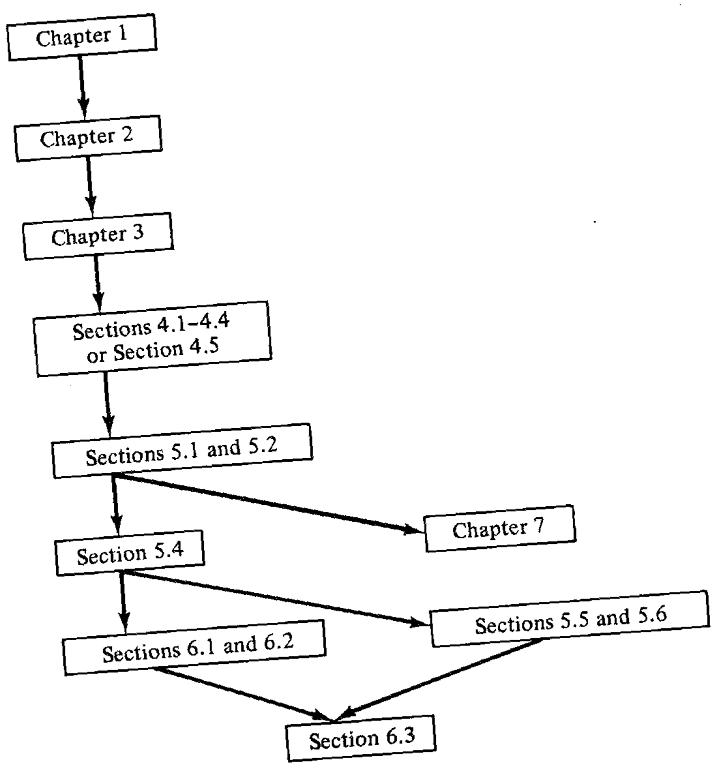
Chapter 5 discusses eigenvalues, eigenvectors, and diagonalization. One of the most important applications of this material occurs in computing matrix limits. We have therefore included an optional section on matrix limits and Markov chains in this chapter even though the most general statement of some of the results requires a knowledge of the Jordan canonical form. Sections 5.4, 5.5, and 5.6 contain material on invariant subspaces, the Cayley-Hamilton theorem, and the minimal polynomial, respectively.

Canonical forms are treated in Chapter 6. Sections 6.1 and 6.2 develop the Jordan form, and Section 6.3 presents the rational form.

Inner product spaces are the subject of Chapter 7. The basic mathematical theory (inner products; the Gram-Schmidt process; orthogonal complements; adjoint transformations; normal, self-adjoint, orthogonal, and unitary operators; orthogonal projections; and the spectral theorem) is contained in Sections 7.1, 7.2, 7.3, 7.5, 7.7, and 7.9. Sections 7.4, 7.6, 7.8, and 7.10 contain diverse applications of the rich inner product structure. The chapter ends with a discussion of bilinear and quadratic forms (Section 7.11).

There are five appendices. The first four, which discuss sets, functions, fields, and complex numbers, respectively, are intended to review basic ideas used throughout the book. Appendix E on polynomials is used primarily in Chapters 5 and 6, especially in Section 6.3. We prefer not to discuss the appendices independently but rather to refer to them as the need arises.

The following diagram illustrates the dependencies among the various chapters.



One final word is required about our notation. Sections denoted by an asterisk (*) are optional and may be omitted as the instructor sees fit. An exercise denoted by the dagger symbol (†) is not optional, however—we use this symbol to identify an exercise that will be cited at some later point of the text.

We are indebted to Douglas E. Cameron, *University of Akron*; Edward C. Ingraham, *Michigan State University*; David E. Kullman, *Miami University*; Carl D. Meyer, Jr., *North Carolina State University*; and Jean E. Rubin, *Purdue University*; who reviewed the entire manuscript, and to our colleagues and students for their suggestions and encouragement while the manuscript was in preparation. Special thanks are due to Jana Gehrke and Marilyn Parmantie for their help in typing the manuscript and to Harry Gaines, Ian List, and the staff of Prentice-Hall for their cooperation during the production process.

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contents

*Sections denoted by an asterisk are optional.

chapter 1

vector spaces

1.1 INTRODUCTION

Many familiar physical notions such as forces, velocities,† and accelerations involve both a magnitude (the amount of the force, velocity, or acceleration) and a direction. Any such entity involving both magnitude and direction is called a vector. Vectors are represented by arrows in which the length of the arrow denotes the magnitude of the vector and the direction of the arrow represents the direction of the vector. In most of the physical situations involving vectors, only the magnitude and direction of the vector are significant; consequently, we shall regard vectors with the same magnitude and direction as being equal irrespective of their positions.

In this section the geometry of vectors will be discussed. This geometry is derived from physical experiments that test the manner in which two vectors interact.

Familiar situations suggest that when two vectors act simultaneously at a point, the magnitude of the resultant vector (the vector obtained by

†The word "velocity" is being used here in its scientific sense—as an entity having both magnitude and direction. The magnitude of a velocity (without regard for the direction of motion) is called its *speed.*

adding the two original vectors) need not be the sum of the magnitudes of the original two. For example, a swimmer swimming upstream at the rate of 2 miles per hour against a current of 1 mile per hour will not progress at the rate of 3 miles per hour. For in this instance the motions of the swimmer and current oppose each other, and the rate of progress of the swimmer is only 1 mile per hour upstream. If, however, the swimmer were moving downstream (with the current), then his rate of progress would be 3 miles per hour downstream.

Experiments show that vectors add according to the following parallelogram law. (See Fig. 1.1.)

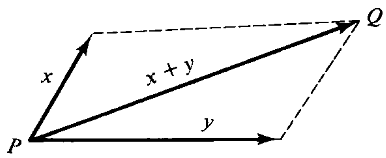

figure 1.1

Parallelogram Law for Vector Addition. The sum of two vectors x *and* y *that act at the same point* P *is the vector in the parallelogram having* x *and* y *as adjacent sides that is represented by the diagonal beginning at* P.

Since opposite sides of a parallelogram are parallel and of equal length, the endpoint Q of the arrow representing $x + y$ can also be obtained by allowing x to act at P and then allowing y to act at the endpoint of x. Likewise, the endpoint of the vector $x + y$ can be obtained by first permitting y to act at P and then allowing x to act at the endpoint of y. Thus two vectors x and y that both act at a point P may be added "tail-to-head"; that is, either x or y may be applied at P and a vector having the same magnitude and direction as the other may be applied to the endpoint of the first—the endpoint of this second vector is the endpoint of $x + y$.

The addition of vectors can be described algebraically with the use of analytic geometry. In the plane containing x and y, introduce a coordinate system with P at the origin. Let (a_1, a_2) denote the endpoint of x and (b_1, b_2) denote the endpoint of y. Then as Fig. 1.2 shows, the coordinates of Q, the endpoint of $x + y$, are $(a_1 + b_1, a_2 + b_2)$. Henceforth, when a reference is made to the coordinates of the endpoint of a vector, the vector should be assumed to emanate from the origin. Moreover, since a vector beginning at the origin is completely determined by its endpoint, we shall sometimes refer to *the point* x rather than *the endpoint of the vector* x if x is a vector emanating from the origin.

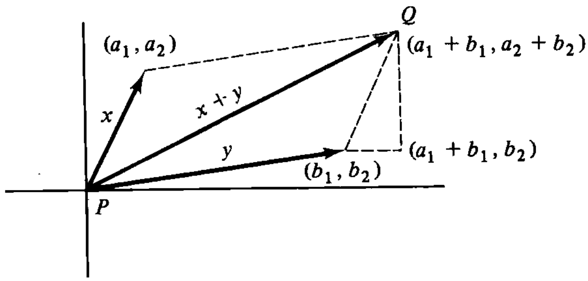

figure 1.2

Besides the operation of vector addition there is another natural operation that can be performed on vectors—the length of a vector may be magnified or contracted without changing the direction of the vector. This operation, called scalar multiplication, consists of multiplying a vector by a real number. If the vector x is represented by an arrow, then for any real number $t \geq 0$ the vector tx will be represented by an arrow having the same direction as the arrow representing x but having length t times the length of the arrow representing x. If $t < 0$, then the vector tx will be represented by an arrow having the opposite direction as x and having length $|t|$ times the length of the arrow representing x. Two non-zero vectors x and y are called *parallel* if $y = tx$ for some non-zero real number t. (Thus non-zero vectors having the same direction or opposite directions are parallel.)

To describe scalar multiplication algebraically, again introduce a coordinate system into a plane containing the vector x so that x emanates from the origin. If the endpoint of x has coordinates (a_1, a_2), then the coordinates of the endpoint of tx are easily shown to be (ta_1, ta_2). (See Exercise 5.)

The algebraic descriptions of vector addition and scalar multiplication for vectors in a plane yield the following properties for arbitrary vectors x, y, and z and arbitrary real numbers a and b:

1. $x + y = y + x$.
2. $(x + y) + z = x + (y + z)$.
3. There exists a vector denoted 0 such that $x + 0 = x$ for each vector x.
4. For each vector x there is a vector y such that $x + y = 0$.
5. $1x = x$.
6. $(ab)x = a(bx)$.
7. $a(x + y) = ax + ay$.
8. $(a + b)x = ax + bx$.

Arguments similar to those given above show that these eight properties, as well as the geometric interpretations of vector addition and scalar multiplication, are true also for vectors acting in space rather than in a plane. We shall use these results to write equations of lines and planes in space.

Consider first the equation of a line in space that passes through two distinct points P and Q. Let O denote the origin of a coordinate system in space, and let u and v denote the vectors that begin at O and end at P and Q, respectively. If w denotes the vector beginning at P and ending at Q, then "tail-to-head" addition shows that $u + w = v$, and hence $w = v - u$, where $-u$ denotes the vector $(-1)u$. (See Fig. 1.3, in which quadrilateral

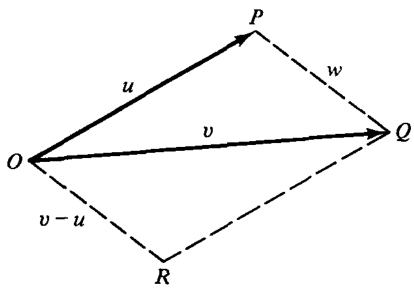

figure 1.3

$OPQR$ is a parallelogram.) Since a scalar multiple of w is parallel to w but possibly of a different length than w, any point on the line joining P and Q may be obtained as the endpoint of a vector that begins at P and has the form tw for some real number t. Conversely, the endpoint of every vector of the form tw that begins at P lies on the line joining P and Q. Thus an equation of the line through P and Q is $x = u + tw = u + t(v - u)$, where t is a real number and x denotes an arbitrary point on the line. Notice also that the endpoint R of the vector $v - u$ in Fig. 1.3 has coordinates equal to the difference of the coordinates of Q and P.

Example. We shall find the equation of the line through the points P and Q having coordinates $(-2, 0, 1)$ and $(4, 5, 3)$, respectively. The endpoint R of the vector emanating from the origin and having the same direction as the vector beginning at P and terminating at Q has coordinates $(4, 5, 3) - (-2, 0, 1) = (6, 5, 2)$. Hence the desired equation is

$$x = (-2, 0, 1) + t(6, 5, 2).$$

Now let P, Q, and R denote any three non-collinear points in space. These points determine a unique plane, whose equation can be found by use of our previous observations about vectors. Let u and v denote the vectors beginning at P and ending at Q and R, respectively. Observe that any point in the plane containing P, Q, and R is the endpoint S of a vector x beginning at P and having the form $t_1u + t_2v$ for some real numbers t_1 and t_2. The endpoint of t_1u will be the point of intersection of the line through P and Q with the line through S parallel to the line through P and R. (See Fig. 1.4.) A similar procedure will locate the endpoint of t_2v.

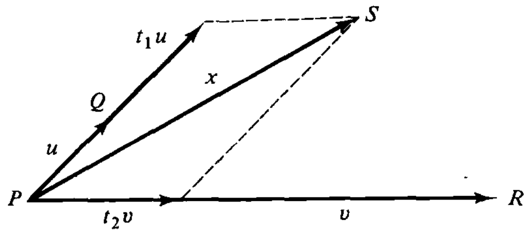

figure 1.4

Moreover, for any real numbers t_1 and t_2, $t_1u + t_2v$ is a vector lying in the plane containing P, Q, and R. It follows that an equation of the plane containing P, Q, and R is

$$x = P + t_1u + t_2v,$$

where t_1 and t_2 are arbitrary real numbers and x denotes an arbitrary point in the plane.

Example. Let P, Q, and R be the points having coordinates $(1, 0, 2)$, $(-3, -2, 4)$, and $(1, 8, -5)$, respectively. The endpoint of the vector emanating from the origin and having the same length and direction as the vector beginning at P and terminating at Q is $(-3, -2, 4) - (1, 0, 2) = (-4, -2, 2)$; likewise the endpoint of the vector emanating from the origin and having the same length and direction as the vector beginning at P and terminating at R is $(1, 8, -5) - (1, 0, 2) = (0, 8, -7)$. Hence the equation of the plane containing the three given points is

$$x = (1, 0, 2) + t_1(-4, -2, 2) + t_2(0, 8, -7).$$

Any mathematical structure possessing the eight properties on page 3 is called a "vector space." In the next section we shall formally define a vector space and consider many examples of vector spaces other than the ones mentioned above.

EXERCISES

1. Determine if the vectors emanating from the origin and terminating at the following pairs of points are parallel.

(a) (3, 1, 2) and (6, 4, 2)
(b) (−3, 1, 7) and (9, −3, −21)
(c) (5, −6, 7) and (−5, 6, −7)
(d) (2, 0, −5) and (5, 0, −2)

2. Find the equations of the lines through the following pairs of points in space.
(a) (3, −2, 4) and (−5, 7, 1)
(b) (2, 4, 0) and (−3, −6, 0)
(c) (3, 7, 2) and (3, 7, −8)
(d) (−2, −1, 5) and (3, 9, 7)

3. Find the equations of the planes containing the following points in space.
(a) (2, −5, −1), (0, 4, 6), and (−3, 7, 1)
(b) (3, −6, 7), (−2, 0, −4), and (5, −9, −2)
(c) (−8, 2, 0), (1, 3, 0), and (6, −5, 0)
(d) (1, 1, 1), (5, 5, 5), and (−6, 4, 2)

4. What are the coordinates of the vector *0* in the Euclidean plane that satisfies condition 3 on page 3? Prove that this choice of coordinates does satisfy condition 3.

5. Prove that if the vector x emanates from the origin of the Euclidean plane and terminates at the point with coordinates (a_1, a_2), then the vector tx that emanates from the origin terminates at the point with coordinates (ta_1, ta_2).

6. Prove that the diagonals of a parallelogram bisect each other.

1.2 VECTOR SPACES

Because such diverse entities as the forces acting in a plane and the polynomials with real number coefficients both permit natural definitions of addition and scalar multiplication that possess properties 1 through 8 on page 3, it is natural to abstract these properties in the following definition.

Definition. *A* vector space (*or* linear space) V over a field† F *consists of a set in which two operations* (*called* addition *and* scalar multiplication, *respectively*) *are defined so that for each pair of elements* x, y *in* V *there is a unique element* x + y *in* V, *and for each element* a *in* F *and each element* x *in* V *there is a unique element* ax *in* V, *such that the following conditions hold:*

†See Appendix C. With few exceptions, however, the reader may interpret the word "field" to mean "field of real numbers" (which we denote by *R*) or "field of complex numbers" (which we denote by *C*).