R. E. BELLMAN, R. E. KALABA AND M. C. PRESTRUD

INVARIANT IMBEDDING & RADIATIVE TRANSFER IN SLABS OF FINITE THICKNESS

INVARIANT IMBEDDING AND RADIATIVE TRANSFER IN SLABS OF FINITE THICKNESS

by

RICHARD E. BELLMAN, ROBERT E. KALABA,
MARCIA C. PRESTRUD

The RAND Corporation

AMERICAN ELSEVIER PUBLISHING COMPANY, INC.

NEW YORK

1963

CONTENTS

Chapter Four
COMPUTATIONAL RESULTS

INTRODUCTION

THE STUDY of radiative transfer is interesting for a variety of physical and mathematical reasons. Not the least of these is its application to the determination of the constitution of planetary atmospheres. This is an inverse problem in which theory and observation must be combined to ascertain unknown cause-and-effect relations and unknown parameters. The analytic results obtained in the study of radiative transfer may also be used in neutron-transport theory because the fundamental equations of the two theories are closely related. Thus, for example, the techniques presented here may be used in the study of shielding and in the design of nuclear reactors.

Even simplified versions of the physical processes occurring in radiative transfer have given rise to a large number of intriguing, difficult mathematical problems requiring a quite sophisticated level of analysis. These results may be found in the books by Hopf [26], Chandrasekhar [21], and Busbridge [20], which contain extensive references.

The classical approach is through the linearized transport equation. However, much difficulty is encountered along the way in establishing existence and uniqueness of solution, and particularly in obtaining a computational procedure.

In a fundamental paper published in 1943, the astrophysicist Ambarzumian [2] presented a radically new approach to the mathematical formulation of these problems, which yielded a new and vastly improved computational treatment for certain geometrical configurations of the medium. This novel and ingenious approach, based on the use of functional equations and physically intuitive principles of invariance, exploited the multistage aspect of the physical processes involved in radiative transfer.

As a result of this pioneering work, new analytic treatments were made available, and simplified computational solutions were obtained.

These ideas were further developed and extensively generalized by Chandrasekhar in a series of fundamental papers and in 1950 in his book cited above. Many otherwise intractable problems were tamed, and great advances were made.

In turn, the "principles of invariance" of Ambarzumian-Chandrasekhar were extended and generalized by Bellman and Kalaba in 1956 and 1957, [9], [10], and applied to the study of inhomogeneous regions of plane, cylindrical, or spherical type, and to stratified regions in general. Results of this nature in an abstract setting were extensively discussed by Preisendorfer [30] and many detailed results have been given by Ueno [40, 41, 42], where other references may be found.

The systematic use of invariance concepts and functional equations in mathematical physics is called *invariant imbedding*. The ideas behind the development given here stem from the theory of dynamic programming on the one hand (Bellman [5], [6]), where the multistage aspect of variational and optimization processes is stressed, and on the other from the generalized theory of branching processes created by Harris, Jánossy, Ramakrishnan, Moyal, and others (Harris [24], [25], and Bellman and Harris [7], [8]), where the "point of regeneration" technique occupies a basic role.

Naturally, a method as basic and natural as that of the use of recurrence, or functional equations, in mathematical physics has many different origins. We can trace it in papers of Rayleigh and Stokes, and it was quite explicitly given in Schmidt [39] in 1907 in a paper that apparently was never followed up. Independently, Redheffer applied invariant-imbedding techniques to study various systems of differential equations arising in scattering problems and to some electromagnetic questions; see Redheffer [36, 37, 38] for a detailed account and references. Many examples also occur in transmission-line theory in connection with ladder networks, and so on, where the use of continued fractions is standard.

In conjunction with Wing, the authors have applied the general methods of invariant imbedding to neutron-transport theory, furnishing new and occasionally improved methods for the studies of shielding and the determination of critical mass of certain geometries.

In a survey paper (Bellman, Kalaba, and Wing [16]), applications of these techniques to scattering and diffusion theory are given. Further work along these lines may be found in the papers of Ramakrishnan [33, 34, 35] and in the works of Redheffer cited above.

In other joint papers, Bellman, Kalaba, and Wing showed how invariant imbedding could be used to reduce two-point boundary-value problems to initial-value problems [17], [18], to show how existence and uniqueness theorems could be obtained on the basis of simple physical-conservation principles [17], to establish connections between classical variational principles and invariant imbedding [19], and, finally, to develop connections between invariant imbedding and the classical transport equation; see Wing [43], Bellman and Kalaba [15], and Mullikin [29].

Parallel to this activity, Bellman and Kalaba showed how invariant imbedding could be used in the study of deterministic and stochastic wave propagation [11, 12, 13, 14]; see also Atkinson [4].

It is certainly important to develop new mathematical formulations of physical processes and new analytic treatments of the fundamental equations. It is also essential to demonstrate that these new techniques actually do furnish efficient computational solutions. Those readers familiar with the idiosyncrasies of computers will confirm the observation that no numerical algorithm can be accepted on faith. Until actual numerical results are obtained, we must withhold judgment as to the efficacy of a method.

Ambarzumian and Chandrasekhar showed that their techniques were computationally effective in the study of semi-infinite plane regions and of homogeneous plane regions of finite thickness. The algorithm that we present is somewhat more direct than these techniques and seems advantageous for the direct computation of the quantities of most physical interest. In this first study we consider isotropic scattering that does not change the state of polarization of the incident flux in plane regions of finite thickness. Although the algorithm we present will handle the case in which both the scattering coefficient and the absorption coefficient vary with the depth in the region, we have concentrated on the case of a homogeneous plane region of finite thickness in order to present a fairly complete analysis of the effect of variation of both thickness and absorption coefficient. To verify the accuracy of our computational results, we compared them with the limiting case of infinite thickness, given by Ambarzumian, and with some results of Chandrasekhar for the finite case. We found complete agreement. In addition, we used slightly different computational techniques and again found agreement. We thus have a fair amount of confidence in the tables we present.

There were several reasons for concentrating initially on the case of isotropic scattering, even though real scattering laws are more complicated. First, in order to validate the computational methods and to decide such questions as what sort of approximations can be used, it is necessary to perform numerous experimental computations, necessitating assumptions that can be handled relatively rapidly and that can be checked against available bench marks. Secondly, the isotropic case has been studied in the literature more extensively than any other case, and it is desirable to have tables against which the various analytic approximations can be checked. We believe that the tables are also of value in that they will assist in estimating the effects of nonisotropy or polarization, after some of the more complicated cases have been computed. Once we abandon the assumption of isotropy, the possible scattering laws are endless and it is desirable to have standards against which deviations can be measured.

Let us also mention the work of Sekera and Viezee [44], which demonstrates how the solution for the flat medium can be used to represent the reflection of parallel sunlight on a spherical planetary atmosphere, and that of Deirmendjian and Clasen [45] on the anisotropy of the scattering of real particles.

The present report is divided into four parts. In the first part, we briefly sketch the physical background of the problem and present the fundamental equations of classical transport theory and of invariant imbedding. The second part contains an explanation of the method used to obtain numerical solutions of the equation furnished by invariant imbedding. Here we use the standard technique of Gaussian quadrature. The third part is devoted to a study of certain analytic properties of the solution as a function of the thickness of the plane region, with particular attention to the limiting behavior of the reflected flux as the finite region expands into the infinite region. Finally, the fourth part presents tables of values of the reflected intensity as a function of the angle of incidence, the angle of reflection, the thickness of the plane-parallel slab, and the ratio of absorption to reradiation. Several typical graphs are given.

We hope at a later time to present computations for more complicated kinds of scattering laws and for cases in which the radiation is time dependent.

We wish to thank Z. Sekera, D. Deirmendjian, and T. E. Harris for a number of illuminating discussions and, particularly the last two, for some very helpful suggestions concerning the text of this monograph.

This study was prepared as part of a program of research undertaken for the Advanced Research Projects Agency by The RAND Corporation.

CHAPTER ONE

THE PHYSICAL MODEL

1. THE RADIATIVE-TRANSFER MODEL

LET US begin by describing the particular physical model of radiative transfer we shall employ in our further analysis. We assume that parallel rays of light of uniform intensity are incident upon an inhomogeneous slab composed of material that absorbs and scatters light. (See Fig. 1.)

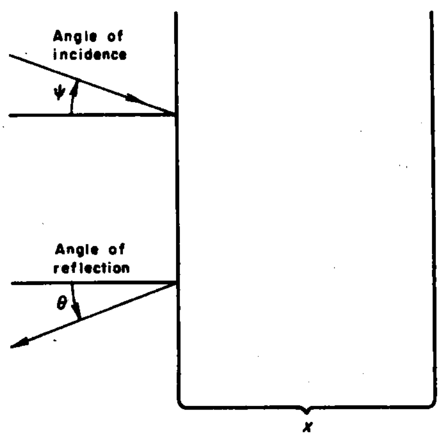

FIG. 1—Incident Rays upon a Plane-parallel Slab of Thickness x

It is desired to determine the intensity of the diffusely reflected light as a function of the intensity of the incident flux, the composition of the slab, and the angles of incidence and reflection.

The properties of the medium composing the slab are the following:

1. In traversing a distance d in any direction in the slab, the intensity I of the beam

is reduced by absorption to $Ie^{-ad} + o(d)$. This relation will be used only for infinitesimal distances, so that we may write this as $I(1 - ad) + o(d)$.

A fraction λ of the absorbed beam is reradiated, and the remaining fraction $1 - \lambda$ is lost.* As a consequence of the inhomogeneity of the medium, the quantities a and λ are dependent, in general, on the part of the slab through which the beam is passing. We assume, however, that they depend only on the depth inside the slab, i.e., that the medium is plane stratified.

2. Radiation is scattered isotropically without changing the polarization. The light that is scattered and reradiated is taken to consist of photons that are treated as if they were point particles. Because of the symmetry of the situation, azimuthal angles may be neglected, and only the angles θ and ψ pictured in Fig. 1 are of significance to the discussion.

2. THE CLASSICAL TRANSPORT EQUATION

For the homogeneous plane-parallel case of finite thickness, the classical transport equation governing densities is

$$\mu \frac{\partial N}{\partial x} + \sigma N = \frac{c}{2} \int_{-1}^{+1} N(x, \mu')\, d\mu'; \tag{2.1}$$

it has as possible boundary conditions

$$\begin{aligned} N(-b, \mu) &= \phi(\mu), \qquad \mu > 0, \\ N(+b, \mu) &= \psi(\mu), \qquad \mu < 0. \end{aligned} \tag{2.2}$$

It is not a trivial matter to obtain a computational solution of this equation. Numerical results have been obtained by Chandrasekhar and may be found in his book. The derivation of the equation is also given there. Our aim here is to present a technique that is equally applicable to the inhomogeneous case, one that has not been previously treated by means of invariance techniques. Further treatment of this equation may be found in papers by Lehner and Wing [27] and by Wing [43].

3. INVARIANT IMBEDDING

In the classical approach, the emphasis is on the internal fluxes, which in the limiting cases—the boundaries—yield the reflected and transmitted fluxes. Here, we reverse the roles and concentrate on the reflected and transmitted fluxes as the primary objects of investiga-

*λ is usually called the *albedo* for single scattering.

tion. It can be shown in several ways that the internal fluxes can be obtained from these quantities; see Bellman, Kalaba, and Wing [16].

Let us introduce the function

(3.1) $\rho(\psi, \theta, x) =$ the specific intensity of reflected radiation, in the direction θ, per unit area on the face of the slab of thickness x, due to a beam of unit intensity incident at angle ψ, the area again being taken parallel to the face of the slab as shown in Fig. 1.

Our aim is now to obtain an equation for this function of the three independent variables θ, ψ, and x. Following the usual procedure of invariant imbedding, as described in Bellman and Kalaba [9], [10], we readily obtain a nonlinear integrodifferential equation for $\rho(\psi, \theta, x)$.

Without going into the analytic details, which may be found in the foregoing references, and for the case of infinite thickness in the books of Ambarzumian and Chandrasekhar, let us briefly indicate the principal idea.

The reflected flux, as noted above, is taken to be a function of the incident angle, the reflected angle, and the thickness x of the slab. After the incident radiation has passed through an infinitesimal slab of thickness Δ, it and the new radiation produced by interaction with scatterers in the infinitesimal slab between x and $x - \Delta$ look into a slab of thickness $x - \Delta$. Hence the reflected radiation from this slab can be described in terms of the same function $\rho(\psi, \theta, x - \Delta)$, where the angles are now dummy variables. Taking account of all interactions yielding terms of lower order than Δ^2, we obtain a differential-integral equation for ρ.

Using these observations, we may write

$$
\begin{aligned}
(3.2) \qquad \rho(\psi, \theta, x + \Delta) = & \left[1 - \frac{a(x)}{\cos\psi}\Delta\right]\rho(\psi, \theta, x)\left[1 - \frac{a(x)}{\cos\theta}\Delta\right] + \frac{a(x)\lambda(x)\Delta}{4\pi\cos\psi} \\
& + 2\pi\frac{a(x)\lambda(x)\Delta}{4\pi\cos\psi}\int_0^{\pi/2}\rho(\psi', \theta, x)\sin\psi'\,d\psi' \\
& + 2\pi\int_0^{\pi/2}\frac{a(x)\lambda(x)\Delta}{4\pi\cos\theta'}\rho(\psi, \theta', x)\sin\theta'\,d\theta' \\
& + (2\pi)^2\int_0^{\pi/2}\rho(\psi, \theta', x)\frac{a(x)\lambda(x)\Delta}{4\pi\cos\theta'}\sin\theta'\,d\theta' \\
& \cdot\int_0^{\pi/2}\rho(\psi', \theta, x)\sin\psi'\,d\psi' + o(\Delta).
\end{aligned}
$$

The terms on the right-hand side of (3.2) arise in the following manner. The first accounts for absorption losses in passing through the layer of thickness Δ on the way in and on the way out. The second is the contribution due to direct scattering from the layer of thickness Δ. The third is due to light that is scattered in the layer of thickness Δ and reflected from the slab extending from 0 to x. The fourth arises from light reflected from the slab extending from 0 to x, and scattered in the slab of thickness Δ. The last, the nonlinear term, represents the contribution of light that is reflected from the slab of thickness x, scattered in the slab of thickness Δ, and rereflected from the slab of thickness x.

Upon passage to the limit we find the equation

$$\begin{aligned}(3.3)\quad \rho_x = {} & \frac{a(x)\lambda(x)}{4\pi\cos\psi} - a(x)\left(\frac{1}{\cos\theta}+\frac{1}{\cos\psi}\right)\rho + \frac{a(x)\lambda(x)}{2\cos\psi}\int_0^{\pi/2}\rho(\psi',\theta,x)\sin\psi'\,d\psi' \\ & + \frac{a(x)\lambda(x)}{2}\int_0^{\pi/2}\rho(\psi,\theta',x)\frac{\sin\theta'}{\cos\theta'}\,d\theta' + \pi a(x)\lambda(x)\int_0^{\pi/2}\frac{\sin\theta'}{\cos\theta'}\rho(\psi,\theta',x)\,d\theta' \\ & \cdot\int_0^{\pi/2}\rho(\psi',\theta,x)\sin\psi'\,d\psi'.\end{aligned}$$

Furthermore, for initial condition we have

$$(3.4)\qquad \rho(\psi,\theta,0)=0.$$

The traditional astrophysical formulation involves the employment of unit areas normal to the direction of flow, rather than parallel to the face of the slab. In addition,

$$(3.5)\qquad u=\cos\psi$$

and

$$(3.6)\qquad v=\cos\theta$$

are used as measures of the angular variables, and the magnitude of the incident intensity is taken to be π. If we set

$$(3.7)\qquad r(v,u,x)=\pi\rho\frac{u}{v},$$

we find that the astrophysical diffuse-reflection function r satisfies the equation

$$\begin{aligned}(3.8)\quad r_x = {} & \frac{a(x)\lambda(x)}{4v} - a(x)\left(\frac{1}{u}+\frac{1}{v}\right)r + \frac{a(x)\lambda(x)}{2}\int_0^1 r(v,u',x)\frac{du'}{u'} \\ & + \frac{a(x)\lambda(x)}{2v}\int_0^1 r(v',u,x)\,dv' + a(x)\lambda(x)\int_0^1 r(v',u,x)\,dv' \\ & \cdot\int_0^1 r(v,u',x)\frac{du'}{u'},\end{aligned}$$

along with the initial condition

(3.9) $$r(v, u, 0) = 0.$$

In order to obtain a function that is symmetric in its angular arguments, we write

(3.10) $$r(v, u, x) = \frac{R(v, u, x)}{4v}.$$

Then it can be seen that the function R satisfies the equation

(3.11) $$\frac{1}{a(x)\lambda(x)}\left[R_x + a(x)\left(\frac{1}{u} + \frac{1}{v}\right)R\right] = \left[1 + \frac{1}{2}\int_0^1 R(v, u', x)\frac{du'}{u'}\right]\left[1 + \frac{1}{2}\int_0^1 R(v', u, x)\frac{dv'}{v'}\right].$$

This is the equation we wish to study. In our computational considerations, we shall assume that

(3.12) $$a(x) \equiv 1,$$

which implies that the mean free path of a particle in the medium is unity. In addition, the thickness x of the medium will be measured in terms of the mean free path, the natural unit of length, and we shall consider only those cases in which

(3.13) $$\lambda = \text{const},$$

though our programs are written for general functions $\lambda(x)$.

4. DISCUSSION

Let us examine the differences between the two formulations. In the classical approach, we are led to a *linear* functional equation (2.1) with a *two-point boundary condition* (2.2). The theory of invariant imbedding leads to a *nonlinear* functional equation with an *initial condition*. Generally speaking, the mechanics of solving differential equations by means of digital computers is such that it is preferable to solve nonlinear initial-value problems, as opposed to linear problems with two-point conditions. The reason for this is that the initial-value problem can be resolved by means of a simple iteration procedure, which is ideally suited to digital computers, whereas the two-point problem requires at some stage the solution of a large system of linear equations. Furthermore, in the course of obtaining the solution for a given thickness, the invariant-imbedding approach automatically grinds out the solution for all intermediate thicknesses, whereas—in the general inhomogeneous case—the conventional approach must be carried out for each thickness. In homogeneous cases, this difficulty can occasionally be overcome (see Chandrasekhar [21], in which invariance principles are used for this purpose). One of the advantages of focusing attention on the reflected flux lies in the fact that the internal fluxes can readily be expressed in terms of these quantities; see Bellman, Kalaba, and Wing [18].

Further discussion of this important use of invariance principles to convert two-point boundary-value problems into initial-value problems will be found in Bellman, Kalaba, and Wing [16], [18]. These references describe the application of the theory of dynamic programming to variational problems.

In the next chapter, we shall discuss different computational approaches to the numerical solution of (3.11).

CHAPTER TWO

COMPUTATIONAL TECHNIQUES

1. GAUSSIAN QUADRATURE

In order to perform an integration by means of a digital computer, it is necessary to reduce the process to an arithmetic operation. This, of course, requires an approximation technique of one type or another. The method we shall employ is one devised by Gauss, which has been used repeatedly with success in neutron-transport theory and radiative transfer theory; see Chandrasekhar [21].

Suppose that the $2N$ numbers a_k, x_k, $k = 1, 2, \cdots, N$ are determined so that the approximate relation

$$\int_{-1}^{1} f(x)\,dx = \sum_{k=1}^{N} a_k f(x_k) \tag{1.1}$$

is *exact* if $f(x)$ is a polynomial of degree $2N - 1$ or less. Choosing $f(x)$ to have the form

$$f(x) = x^r P_N(x), \tag{1.2}$$

for the N different values $r = 0, 1, \cdots, N-1$, where $P_N(x)$ is the Legendre polynomial of degree N, it is easy to see that $P_N(x_k) = 0$ for $k = 1, 2, \cdots, N$. Consequently, the x_k must be the roots of the Nth Legendre polynomial.

Choosing the polynomial $f(x)$ to be 0 at $x = x_1, x_2, \cdots, x_N$, except at x_k, where it has the value 1, namely

$$f(x) = \frac{P_N(x)}{(x - x_k)P'_N(x_k)}, \tag{1.3}$$

we obtain the value

$$\int_{-1}^{1} \frac{P_N(x)\,dx}{(x - x_k)P'_N(x_k)} = a_k, \qquad k = 1, 2, \cdots, N. \tag{1.4}$$

These numbers, a_k, which can be evaluated in simpler terms, are called the *Christoffel numbers.*

They have been extensively tabulated and can be readily generated by means of simple algorithms.

A simple change of variable enables us to use a similar formula for any finite interval $[a, b]$ in place of $[-1, 1]$. In particular, we shall use the interval $[0, 1]$.

2. ZEROS OF LEGENDRE POLYNOMIALS; CHRISTOFFEL NUMBERS

Table 1 gives the zeros $\{x_k\}$ of the Legendre polynomial $P_n(x)$, $n = 1, 2, \cdots, 12$, and the corresponding weights $\{a_k\}$ for Gauss' mechanical quadrature formula (1.1). Since $x_k = -x_{n-k+1}$ and $a_k = a_{n-k+1}$, only half of the values are tabulated.

Table 1

ZEROS x_k OF LEGENDRE POLYNOMIALS; CHRISTOFFEL NUMBERS a_k

n	k	x_k	a_k
2	1	0.577350269189626	1.000000000000000
3	1	0.774596669241483	0.555555555555556
	2	0.000000000000000	0.888888888888889
4	1	0.339981043584856	0.652145154862546
	2	0.861136311594053	0.347854845137454
5	1	0.538469310105683	0.478628670499366
	2	0.906179845938664	0.236926885056189
	3	0.000000000000000	0.568888888888889
6	1	0.238619186083197	0.467913934572691
	2	0.661209386466265	0.360761573048139
	3	0.932469514203152	0.171324492379170
7	1	0.405845151377397	0.381830050505119
	2	0.741531185599394	0.279705391489277
	3	0.949107912342759	0.129484966168870
	4	0.000000000000000	0.417959183673469
8	1	0.183434642495650	0.362683783378362
	2	0.525532409916329	0.313706645877887
	3	0.796666477413627	0.222381034453374
	4	0.960289856497536	0.101228536290376
9	1	0.324253423403809	0.312347077040003
	2	0.613371432700590	0.260610696402935
	3	0.836031107326636	0.180648160694857
	4	0.968160239507626	0.081274388361574
	5	0.000000000000000	0.330239355001260

Table 1—continued

n	k	x_k	a_k
10	1	0.148874338981631	0.295524224714753
	2	0.433395394129247	0.269266719309996
	3	0.679409568299024	0.219086362515982
	4	0.865063366688985	0.149451349150581
	5	0.973906528517172	0.066671344308688
11	1	0.269543155952345	0.262804544510247
	2	0.519096129206812	0.233193764591990
	3	0.730152005574049	0.186290210927734
	4	0.887062599768095	0.125580369464905
	5	0.978228658146057	0.055668567116174
	6	0.000000000000000	0.272925086777901
12	1	0.125233408511469	0.249147045813403
	2	0.367831498998180	0.233492536538355
	3	0.587317954286617	0.203167426723066
	4	0.769902674194305	0.160078328543346
	5	0.904117256370475	0.106939325995318
	6	0.981560634246719	0.047175336386512

3. APPROXIMATE SYSTEM OF DIFFERENTIAL EQUATIONS

By means of the quadrature formula

$$\int_0^1 g(w)\,dw = \sum_{k=1}^{N} c_k g(w_k), \tag{3.1}$$

where the c_k and w_k are determined from the foregoing quadrature formula through a simple change of scale, equation (3.11) of Chapter 1 is replaced by the approximate equation

$$\begin{aligned} \frac{1}{\lambda(x)}\left[\frac{\partial R(v,u,x)}{\partial x} + \left(\frac{1}{u}+\frac{1}{v}\right)R(v,u,x)\right] &= 1 + \frac{1}{2}\left[\sum_{k=1}^{N}\frac{c_k}{w_k}R(w_k,u,x)\right] + \frac{1}{2}\left[\sum_{k=1}^{N}\frac{c_k}{z_k}R(v,z_k,x)\right] \\ &\quad + \frac{1}{4}\left[\sum_{k=1}^{N}\frac{c_k}{w_k}R(w_k,u,x)\right]\left[\sum_{k=1}^{N}\frac{c_k}{z_k}R(v,z_k,x)\right], \end{aligned} \tag{3.2}$$

with the initial condition

$$R(v,u,0) = 0. \tag{3.3}$$