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OPERATOR ALGEBRAS AND GROUP REPRESENTATIONS VOLUME I

Edited for INCREST by
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Operator Algebras and Group Representations

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Volume I

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Foreword

From the 1st to the 13th of September 1980, the Department of Mathematics of the National Institute for Scientific and Technical Creation organized an International Conference on Operator Algebras and Group Representations held in Neptun on the Romanian Black Sea Coast.

The research contracts between the Mathematics Department of INCREST and the National Council for Science and Technology constituted the generous framework which made possible the organization of this conference. The conference also benefited from the cooperation of the Romanian Academy and of the Mathematics Department of the University of Bucharest.

The conference was attended by a large number of mathematicians coming from both fields involved, which, in our opinion, confirms the interest of specialists in operator algebras and group representations in having joint meetings. We believe that this was the main source of the success of the conference.

These two volumes contain texts of invited addresses as well as contributions of participants accepted on the basis of referees' reports.

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Volume II

Factorization in $L^p(G)$ and the Plancherel theorem for non-unimodular Lie groups

J. F. Aarnes

1. Introduction

In the last decade a considerable amount of work has been done concerning the establishment of Plancherel theorems for non-unimodular groups, and the actual computation of the Plancherel measures for specific Lie groups (cf. the papers of Kleppner and Lipsman [11], [12], Duflo and Moore [7], Keene, Lipsman and Wolf [10], Lipsman and Wolf [13], Moore [14], Penney [15], Pukanszky [17], Tatsuuma [18] and others). The main result of the present paper, Theorem 5.1, is a sharpened version of the Plancherel theorem which appeared in [13] for type I Lie groups. More specifically we show that it is possible to choose a finite positive measure on G and a semi-invariant operator D of weight $\Delta^{-1/2}$ in $L^2(G)$ which is positive, self-adjoint, invertible and associated with $\mathcal{R}(G)$, the right von Neumann algebra of G, such that $\operatorname{Dom} D \supseteq C_c^\infty(G)$ and that D enters into the Plancherel formulas together with the chosen measure. In turn this enables us to establish the validity of the formulas for functions belonging to $C_c^\infty(G)$ (and even larger spaces which are dense in $L^2(G)$). We also show that any operator D with the properties above is necessarily given by a distribution μ (i.e. $D\varphi = \varphi * \mu$) and that μ is the distributional derivative of an L^2-function. Our methods are quite different from those of [7], [11] and [13] and rely heavily on elliptic regularity theory for differential operators. One of our main tools is the factorization result which appears in Theorem 3.1. Roughly it says that there is a fixed function $\xi_0 \in L^1(G) \cap L^2(G)$ such that any function f in $L^P(G)$ which is sufficiently differentiable may be written as a convolution product $f = \xi_0 * f'$, where $f' \in L^P(G)$. The paper is organized as follows. Section 2 contains preliminary material and introduces certain spaces $\mathcal{A}_p \subseteq L^P(G)$ which plays an important role in the later parts of the paper. Section 3 contains the factorization results and an outline of the proofs. Section 4 is devoted to a discussion of the relationship between semi-invariant operators, intertwining forms and distributions. No proofs are given here. In the final Section 5 we present our version of the global Plancherel theorem for Lie groups of type I together with a sketch of its proof.

We adopt the following conventions. Integration on G will be with respect to a fixed left Haar-measure. The modular function Δ is given by

$$\int_G f(yx)\,dy = \Delta(x^{-1}) \int f(y)\,dy.$$

We define $f^*(x) = \overline{f(x^{-1})}\,\Delta(x)^{-1}$, $f^\natural(x) = \overline{f(x^{-1})}\,\Delta(x)^{-1/2}$ and $\tilde{f}(x) = \overline{f^*(x)}$. If μ is a distribution we put $\langle \bar{\mu}, \varphi \rangle = \overline{\langle \mu, \bar{\varphi} \rangle}$ and $\langle \check{\mu}, \varphi \rangle = \langle \mu, \tilde{\varphi} \rangle$; $\varphi \in C_c^\infty(G)$.

2. The spaces $\mathcal{A}_p$ $(1 \le p < \infty)$

Let G be a Lie group which is countable at infinity. Let $\mathfrak{g}$ denote its Lie algebra and let $\mathcal{G}$ be the complexification of the universal enveloping algebra of $\mathfrak{g}$. Let $D_r(G)$ (resp. $D_l(G)$) denote the algebra of right (resp. left) invariant differential operators on G. If $X \in \mathfrak{g}$ we also let $\tilde{X}$ denote the corresponding right invariant vector-field on G, while X denotes the corresponding left invariant vector-field. Explicitly, for $f \in C^\infty(G)$, $X \in \mathfrak{g}$:

$$(Xf)(x) = \lim_{t \to 0} \frac{1}{t} \{f(\exp(-tX)x) - f(x)\},$$

$$(\tilde{X}f)(x) = \lim_{t \to 0} \frac{1}{t} \{f(x \exp tX) - f(x)\}.$$

Let $X_1, \ldots, X_d$ be any basis for $\mathfrak{g}$, and let $\alpha = (\alpha_1, \ldots, \alpha_d)$; $\alpha_i \in \mathbb{N}$, $i = 1, \ldots, d$ be any multi-index. Elements of the form $X^\alpha = X_1^{\alpha_1} \cdots X_d^{\alpha_d}$ (resp. $\tilde{X}^\alpha = \tilde{X}_1^{\alpha_1} \cdots \tilde{X}_d^{\alpha_d}$) will then constitute a basis for $D_r(G)$ (resp. $D_l(G)$). Accordingly, if D is any element in $\mathcal{G}$ we let D (resp. $\tilde{D}$) denote the corresponding element in $D_r(G)$ (resp. $D_l(G)$).

Let $D_*(G)$ denote the algebra of differential operators on G generated by $D_r(G)$ and $D_l(G)$. Since all elements in $D_r(G)$ commute with all elements in $D_l(G)$ we have

$$D_*(G) = [RL : R \in D_r(G), L \in D_l(G)].$$

We now choose, once and for all, a left Haar-measure on G and let $L^P(G)$ $(1 \le p < \infty)$ have the standard meaning with respect to this measure.

2.1. *Definition.* The space $\mathcal{A}_p$ is given as all functions $f \in C^\infty(G)$ such that $Mf \in L^P(G)$ for all $M \in D_*(G)$. For each $M \in D_*(G)$ we define $\|f\|_M = \|Mf\|_p$ $(f \in \mathcal{A}_p)$ and equip $\mathcal{A}_p$ with the topology derived from the family of seminorms $\{\|\cdot\|_M ;\ M \in D_*(G)\}$.

Our first objective is to give an alternative description of the space $\mathcal{A}_p$ and its topology. The left and right regular representation of G in $L^P(G)$ are given by

$$[\lambda_p(x)f](y)=f(x^{-1}y),$$
$$[\rho_p(x)f](y)=f(yx)\,\Delta(x)^{1/p}$$

($f\in L^P(G)$, $x, y\in G$). The representation σ_p of $G\times G$ in $L^P(G)$ is then given by $\sigma_p(x, y)=\lambda_p(x)\rho_p(y)$, $x, y\in G$. λ_p, ρ_p and σ_p are continuous and norm-preserving representations, unitary if $p=2$. For any representation γ of G let $C^\infty(\gamma)$ denote the space of differentiable vectors for γ. From Goodman's result [8], Theorem 1.1, it easily follows that $C^\infty(\sigma_p)=C^\infty(\lambda_p)\cap C^\infty(\rho_p)$. For any representation γ of G we also denote by γ its infinitesimal form, i.e. the representation of the Lie algebra and given by

$$\gamma(X)\xi=\frac{d}{dt}\gamma(\exp tX)\xi|_{t=0}$$

($X\in \mathfrak{g}$, $\xi\in C^\infty(\gamma)$). $C^\infty(\gamma)$ is a common dense domain for the operators $\gamma(X)$ and γ has an extension to a representation of $\mathcal{G}$ on $C^\infty(\gamma)$, which again is denoted by γ. Now, by Poulsen's results [16], p. 114, we have the following characterizations of $C^\infty(\lambda_p)$ and $C^\infty(\rho_p)$:

$$C^\infty(\lambda_p)=\{f\in C^\infty(G): Df\in L^P(G), D\in\mathcal{G}\},$$
$$C^\infty(\rho_p)=\{f\in C^\infty(G): \tilde{E}f\in L^P(G), E\in\mathcal{G}\}.$$

The topology of $C^\infty(\lambda_p)$ (resp. $C^\infty(\rho_p)$) is given by the family of seminorms $\|Df\|_p$ (resp. $\|\tilde{D}f\|_p$), $D\in\mathcal{G}$, under which both spaces are complete, locally convex and metrizable, i.e. they are Fréchet spaces.

By a slight variation of Poulsen's arguments (cf. [16], Lemmas 5.1 and 5.2) we get the following Sobolev lemma.

2.2. Lemma. *Let $d=\dim G$ and let s be an integer satisfying $s>d/p$ ($1\le p<\infty$). For each compact neighbourhood of the identity $e\in G$ there is a positive constant C such that for any $x\in G$:*

(i) $|f\,\Delta^{1/p}(x)|\le C\sum_{|\alpha|\le s}\{\int_{x^{-1}U}|X^\alpha f(y)|^p\,dy\}^{1/p}$ *for all* $f\in C^\infty(\lambda_p)$,
(ii) $|f(x)|\le C\sum_{|\alpha|\le s}\{\int_{Ux}|\tilde{X}^\alpha f(y)|^p\,dy\}^{1/p}$ *for all* $f\in C^\infty(\rho_p)$.

For $f\in L^P(G)$ we put $f^\Delta(x)=f(x^{-1})\,\Delta(x^{-1})^{1/p}$. Then $\|f^\Delta\|_p=\|f\|_p$ and $f^{\Delta\Delta}=f$. For any differential operator M we now define $M^\Delta f=(Mf^\Delta)^\Delta$ ($f\in C^\infty(G)$). Then $M\to M^\Delta$ is an algebra isomorphism of $D_r(G)$ onto $D_l(G)$ and vice versa. We have $M^{\Delta\Delta}=M$ and $M\to M^\Delta$ is an automorphism of $D_*(G)$. The map $f\to f^\Delta$ is a linear topological isomorphism of $C^\infty(\lambda_p)$ onto $C^\infty(\rho_p)$.

On the basis of the results above it is now a fairly straightforward matter to show the following:

2.3. Proposition. *$\mathcal{A}_p$ is a Fréchet space and coincides with $C^\infty(\sigma_p)$. The elements of $\mathcal{A}_p$ are bounded and vanish at infinity. $f \to f^\Delta$ is a linear topological isomorphism of $\mathcal{A}_p$ onto itself. $\mathcal{A}_p$ is invariant under left and right translations (which are continuous, even differentiable operations). If $f \in \mathcal{A}_1$, $g \in \mathcal{A}_p$ then $f * g \in \mathcal{A}_p$ and the map $(f, g) \to f * g$ is jointly continuous of $\mathcal{A}_1 \times \mathcal{A}_p$ into $\mathcal{A}_p$, i.e. $\mathcal{A}_p$ is a continuous left $\mathcal{A}_1$-module. In particular $\mathcal{A}_1$ is an involutive Fréchet algebra under the usual involution $f \to f^*$. If $p \leq q$ we have continuous dense imbeddings*

$$C_c^\infty(G) \to \mathcal{A}_p \to \mathcal{A}_q \to L^q(G).$$

In some sense $\mathcal{A}_p$ is the largest space of differentiable functions in $L^P(G)$ it is reasonable to consider. Let us say that a locally convex space $\mathcal{F}$ is an *$\mathcal{A}$-space of type p* if the following conditions are satisfied:

(i) $\mathcal{F} \subseteq L^P(G) \cap C^\infty(G)$ and the injection of $\mathcal{F}$ into $L^P(G)$ is continuous.

(ii) $\mathcal{F}$ is invariant under $D_*(G)$ and each $M \in D_*(G)$ is continuous on $\mathcal{F}$.

It is then easily seen that if $\mathcal{F}$ is an $\mathcal{A}$-space of type p then $\mathcal{F} \subseteq \mathcal{A}_p$ and the injection is continuous. Before we turn to applications of this we offer one more definition. Let us say that a distribution μ is *strongly tempered* if μ is continuous on $C_c^\infty(G)$ with respect to the relative topology of $\mathcal{A}_1$.

2.4. Proposition. *Let μ be a distribution of positive type on G. Then μ is strongly tempered.*

Proof. By Theorem 3.1 of [2] there exist a continuous function f of positive type on G and an element $M \in D_*(G)$ such that $\mu = Mf$. Hence, if $\varphi \in C_c^\infty(G)$:

$$|\langle \mu, \varphi \rangle| = |\langle f, M\varphi \rangle| \leq f(e) \, \|M\varphi\|_1$$

which proves the assertion.

2.5. Corollary. *Suppose G is semisimple and that μ is a distribution of positive type on G. Then μ is tempered in the sense of Harish-Chandra* (cf. [9]).

Proof. $\mathcal{C}_1$ is an $\mathcal{A}$-space of type 1.

This result has also been obtained by Barker [4]. Later on in this paper we shall need still another characterization of the topology of $\mathcal{A}_p$ which