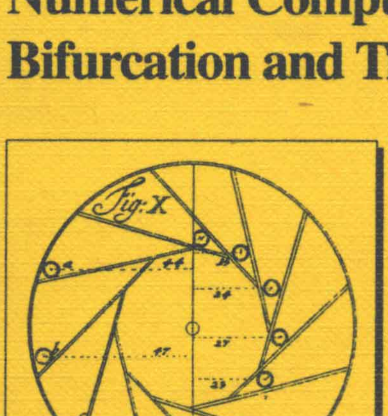
Lecture Notes in Mathematics 1483

Eduard Reithmeier

Periodic Solutions of Nonlinear Dynamical Systems

Numerical Computation, Stability, Bifurcation and Transition to Chaos



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for the publication of monographs

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§ 3. Final manuscripts should be in English. They should contain at least 100 pages of scientific text and should include

- a table of contents;
- an informative introduction, perhaps with some historical remarks: it should be accessible to a reader not particularly familiar with the topic treated;
- a subject index: as a rule this is genuinely helpful for the reader.

Further remarks and relevant addresses at the back of this book.

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Preface

Watching the cyclic motion of the planets around the sun, the seasons on earth, our biological day and night rythm, the periodicity of life itself (to be born, live and die) are only a few examples showing that we are embedded in and surrounded by cyclic phenomena. In fact, periodic solutions of mathematical models of physical systems arise precisely because we live in a cyclic (or rly so) world.

In studying the extensive literature on almost periodic solutions of non-linear ODE's, I think the first questio g to mind is why there should be another text on this topic. My reasons for w g are multiple. I suppose anyone who has ever studied the theory of non-linear ations was surprised to discover motion readily discernible by us, the periodi tion, is so difficult to investigate analytically despite highly developed math tools; and in most cases there is no analytical access at all. There is a lar of mathematicans and scientists who developed analytical tools for the investig n of periodic solutions of non-linear ODE's; indeed, the field is nearly saturated. Hence, the probability is very low of making serious progress in developing further analytical methods. However, the numerical treatment of ODE's reached a very high level in the last two decades, and it seems reasonable to apply and to tailor numerical algorithms for the purpose of computation and investigation of periodic solutions. Furthermore, the bifurcation behavior of periodic "modes", due to varying parameters of a dynamical system, was found to be the most important mechanism to explain and to investigate transition into "chaos". Bifurcation of cyclic motions is caused by destabilization and therefore computation, stability and bifurcation analyses of periodic solutions are elementary steps in gaining essential information about a non-linear dynamical system.

Although periodic motions are related to many applications, most of the literature about this topic involves a very abstract mathematical framework. My intention was to connect mathematical framework and applications. The mathematical tools I employ, are based on modern applied mathematics and numerical analysis. Since this book is based primarily on work undertaken during my research activities at "Institut B für Mechanik, TU München", I was mainly concerned with technical problems.

Technical problems lead in many cases to mathematical descriptions which involve dicontinuities with respect to the state space variables of the vector field. Aside from some trivial exceptions, periodic solutions of dynamical systems with discontinuities have never really been closely investigated, either in application or in theorry. Therefore I found it useful to investigate these systems with same intensity in the form of differentiable dynamical systems.

I would like to mention that this book would never have been completed in this form if the director of the Institut B für Mechanik, Prof. Dr.-Ing. F. Pfeiffer had not provided his interest and support throughout the whole time of my stay. Particularly, his liberal attitude made it possible to investigate interdisciplinary topics like the ones discussed here. Furthermore, I am thankful for the influence of his style of research which encouraged and motivated me constantly. I also wish to thank Prof. Dr. rer. nat. R. Bulirsch for reviewing the manuscript and for his support in connecting modern applied mathematics with engineering problems. I would like to express my special thanks to Prof. Dr. rer. nat. P. Rentrop for his useful and constructive suggestions on the treatment of numerical problems. Regarding the English version I thank Prof. Dr. Dr. hc. mult. G. Leitmann for his suggestions and influence during my stay at UC Berkeley in improving the readability of this text. In addition, my thanks are due to all of my collegues who supported the work in some sense, in particular Dr.-Ing. habil. H. Bremer, Dr.-Ing. K. Karagiannis, Dipl.-Ing. K. Richter and Dipl.-Ing. A. Kunert who read the manuscript carefully. Last but not least it is a pleasure for me to mention my appreciation for the excellent work of Monika Böhnisch who typed the manuscript and of Monika Rotenburg who reviewed the English version of the manuscript.

Berkeley, March 1991 — Eduard Reithmeier

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Contents

1 Introduction

... denn das reine Chaos ist vollkommen uninteressant.
W. Heisenberg [1]

1.1 Motivation and objective

The determination and investigation of singular points and periodic solutions of non-linear vibrating systems is of theoretical interest as well as of technical importance. The theoretical interest stems from the convergence of many demanding mathematical fields such as the theory of fixed points and singularities, the theory of non-linear algebraic equations or the theory of ODE's. The technical relevance is shown in several points: limit cycles occur in numerous areas of application and – because of their structural invariant properties – they play an important role in the behavior of the non-linear vibrating system. Last but not least, the bifurcation of a limit cycle is one of the main reasons for the genesis of the irregular behavior of the system.

The dynamics of machinery is one of the main areas of application. Belts and chains for drive and control, for instance, rotate in a stable and continuous way if the number of revolutions is low. By increasing the speed the motion loses its stability and a periodic vibration will result. The same effect occurs in railway vehicle systems that have a stable straight-forward run at low speeds. If the vehicle surpasses a certain velocity, a limit cycle motion will result. This phenomen is not restricted to differentiable systems. Systems which are discontinuous with respect to state space variables also show this behavior. Examples for this case are damping elements with dry friction which are implemented between the blades of a turbine. If the number of revolutions is low, the damping elements lock. With increasing speed, however, a limit cycle vibration occurs. Another example with discontinuities in the state space are non-loaded gear wheel sets in multistage gear boxes. By increasing the speed or the amplitude of excitation or the damping, which is caused by oil, the vibration behavior is characterized by a series of bifurcations of periodic solutions.

A further interesting area of application is a well-known effect: amplifiers or microphones begin to make a whistling sound if the feedback is sufficiently high. This sound is equivalent to a limit cycle motion. Furthermore, many applications from other areas such as biology, laseroptics, quantuum-mechanics or celestial-mechanics show the occurance of limit cycle vibrations.

Limit cycle motions have been a well-known phenomen of the non-linear vibration theory for many years [POINCARÉ 1912], [HOPF 1942], [MAGNUS 1955]. The

[1] Werner Heisenberg: Schritte über Grenzen. Serie Piper, vol. 336, p.236.

systems are mainly investigated by approximation theory concerning this aspect. In most cases applications have been restricted to systems with one degree of freedom. Moreover, these systems have been investigated without discontinuities.

Hence, the targets of this book are:

1. to develop techniques or to modify existing methods to compute fixed points and periodic solutions of non-linear vibrating systems,

2. to determine criteria and resulting methods which can be used to investigate the stability- and bifurcation behavior of the numerically computed periodic solutions,

3. to obtain a connection between the periodic solutions and the global behavior of the system from the theory of normal forms of POINCARÉ and SIEGEL, and to acquire knowledge of the stability- and bifurcation behavior.

The contents of this book are divided in two main chapters: the first part is dealing with differentiable systems having no discontinuities with respect to the state space variables (chapter 2) and the second part is dealing with differentiable systems with a finite number of discontinuities (chapter 3). In both chapters the main emphasis of application lies on dynamical systems.

From the mathematical point of view fixed points are also periodic solutions, namely the trivial constant solutions with respect to time. On the other hand, periodic solutions are fixed points in a suitable POINCARÉ–section in the phase space. Therefore, it is best to begin the investigations with the classification of the fixed points or the singularities, respectively. There is an immediate connection between the singularities and the theory of normal forms of POINCARÉ and SIEGEL. Furthermore, the theory of normal forms is appropriate to create connections between the stability and the bifurcation of singularities and the diffeomorphic transformation of the nonlinear system into the system linearized around the singularity. For these reasons in chapter 2.3 the essential parts of the theory of normal forms will be presented. In particular, this theory will be adapted to dynamical systems. Chapter 2.4 deals with practical aspects of classifying singularities. The classification of these singularities is demonstrated in a series of examples.

HAMILTONian systems belong to dynamical systems which have a great variety of mathematical structures. These systems are neither dissipative nor internally or externally excited. This is the reason why HAMILTONian systems play a special role and have to be treated in a separate way, which is done in chapter 2.5. As an example for HAMILTONian systems, the double pendulum is investigated. For this example in chapter 2.5.4 all periodic solutions will be computed, which are necessary to explain the irregular behavior of such systems in certain energy intervals (cf. chapter 2.8.4).

Dynamical systems which are dissipative and excited as well have less mathematical structure, but they often occur in technical problems. In such systems the limit cycle motion is dominant. Methods (such as the multiple shooting method) for computing limit cycles can be found in chapter 2.6. The efficiency of the methods will be demonstrated by a railway vehicle system with four degrees of freedom.

In many areas of application it is important to vary the system parameters to obtain periodic solutions with additional properties. One possibility is to construct a periodic solution with a minimal time of period or maximal stability. Another example is the so-called "synchronisation". By this method an excitable vibrating system will be adjusted to the frequency of a harmonic outside excitation. In chapter 2.7 a numerical algorithm, consisting of the HAMILTONian theory of optimization and the multiple shooting method, will be constructed to compute periodic solutions for this problem. The algorithm will be applied to the double pendulum and the railway vehicle system. Furthermore, a connection between the numerical computation of bifurcation points and this algorithm can be established.

Chapter 2.8 deals with the stability and bifurcation of periodic solutions. It will be shown that each solution embedded in a field of asymptotic stable solutions must converge to a limit function. This limit function is either a fixed point or a limit cycle. Two necessary conditions will be obtained for the bifurcation of periodic solutions. As above, the results are applied to the model of a railway vehicle system and double pendulum.

All results of chapter 2 can be transferred with some modifications to differentiable systems with discontinuities. Therefore, chapter 3 is divided into the same subjects as chapter 2. However, the treatment of the corresponding parts is shorter. For the application of the methods, a one-staged gear wheel set and an excitation model with dry friction will be taken into consideration.

1.2 Survey of literature

1.2.1 Existence of periodic solutions

The interest in periodic solutions of non-linear vibrating systems goes back to the beginning of this century. Already [POINCARÉ 1912] investigated periodic solutions of non-linear dynamical systems. In connection with the restricted three-body-problem he studied fixed points of area-preserving one-to-one transformations of simply connected areas on the plane. However, he could not prove his well known "last geometric problem" himself. BIRKHOFF solved this problem some years later and extended it to an arbitrary dimension of the state space. Based on this theorem, [BIRKHOFF, LEWIS 1933] have proved the existence of an infinite number of periodic solutions of a conservative system in the neighborhood of a known periodic solution, which could also be a fixed point of the "general stable type".

An interesting step was taken by [SEIFFERT 1945] twelve years later. He showed, that for each energy level of a dynamical system, described by the LAGRANGEian equations $\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right)^{\cdot} - \frac{\partial T}{\partial \mathbf{q}} + \frac{\partial V}{\partial \mathbf{q}} = \mathbf{0}$, at least one periodic solution exists. In contrast to the theorem of BIRKHOFF, his proof was based purely on the tools of differential geometry. He was looking for closed geodesics on a RIEMANNian manifold with the metric $d\,s^2 := (H-V)d\mathbf{q}^T\mathbf{M}\,d\mathbf{q}$, where $\mathbf{M}$ is the mass matrix, H the HAMILTONian function and V the potential function of the system.

Parallel to these results, [HOPF 1942] proved a theorem which is very useful for technical problems. This theorem supplies a sufficient criterion for the existence of limit cycles in the neighborhood of a singularity of a dissipative and excited non-linear vibrating system.

In the following years mainly [SIEGEL 1954], [SIEGEL 1971] and [MOSER 1953] were occupied with a series of extensions of the theorem of BIRKHOFF and the so-called "resonant case".

Later in the 1960's and '70's many authors such as [HARRIS 1966], [BERGER 1971], [GORDON 1971], [RABINOWITZ 1978a], [EKELAND 1979], [AMANN, ZEHNDER 1980] etc., dealt with periodic solutions of HAMILTONian systems. For special classes of these systems they obtained theorems about the existence of periodic solutions. The main idea of these proofs was to transform the problem of finding periodic solution into a minimax problem of the calculus of variation. By doing this, it turns out that for each periodic solution a critical point of the corresponding variational problem exists.

Other approaches – coming from the differential topology – were made by [MARKUS 1960], [FULLER 1967] and [WEINSTEIN 1973]. Analogously to the topological index in the theory of fixed points, FULLER defined an index for periodic solutions of autonomous systems and applied it to prove the existence of periodic solutions of HAMILTONian systems. In 1973 WEINSTEIN showed that on each energy surface of a HAMILTONian system $\dot{\mathbf{z}} = \mathbf{J}\cdot DH(\mathbf{z})$ ($\mathbf{J}$ symplectic) at least f periodic solutions exist which are seperated. The energy surface lies in a certain neighborhood of a singularity $\mathbf{z}_0 \in \mathbb{R}^{2f}$ and furthermore the HESSE–matrix $D^2H(\mathbf{z}_0)$ must be positive definite. [MOSER 1976] extended this result to arbitrary systems $\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z})$ which have a first integral.

A survey of the existence-statements of periodic solutions of non-linear dynamical systems may be found in [YOSHIZAWA 1975], [RABINOWITZ 1982] and [DUISTERMAAT 1984] for instance.

1.2.2 Numerical computation of periodic solutions

An analytical expression of a periodic solution is mostly restricted to trivial cases. Therefore, up to the 60's the investigations were concentrated on the existence or uniqueness, or on the approximation methods (cf. [MAGNUS 1955]). Since POINCARÉ published his results periodic solutions have mainly been used in the field of pure mathematics. Also the early numerical treatment by the analogue-computer for the DUFFING–oscillator, the VAN DER POL–oscillator or other examples with one degree of freedom, did not change the situation.

A new era has started in the 80's: numerical algorithms were employed to compute and investigate periodic solutions. [SEYDEL 1983] as well as [HOLODNIOK, KUBIČEK 1984] used the multiple shooting method developed by [BULIRSCH, STOER, DEUFLHARD 1977]. The idea is based on finding the periodic solution by solving a boundary value problem with an unknown time of period. Because of the non-uniqueness of the periodic solutions in the autonomous case, it is problematic to formulate the boundary-value problem. Therefore – based on the multiple shooting method – [DEUFLHARD 1984] generated a modified GAUSS–NEWTON–technique to get rid of this problem.

Another interesting proposal – based on the index theory of POINCARÈ–BENDIXON – came from [HSU 1980a,b]. The idea is to generate a POINCARÈ–mapping $P : \sum \rightarrow \sum$ ($\sum \subset TM$, $codim \sum = 1$) in the state space TM. Then the periodic solution is equivalent to a fixed point of P. Now, if a JORDAN–curve $\Gamma : S^{n-1} \rightarrow \sum$ is continuously deformed, the index of $P - id_{\sum}$ changes from $+1$ to -1 or vice versa if the curve Γ passes a fixed point of P.

1.2.3 Bifurcation and stability of periodic solutions

The investigation of stability and bifurcation requires the periodic solutions either in analytical or numerical form. The first investigations in this area were of pure analytical nature. Especially the criteria of [HOPF 1955] should be mentioned. They suppply a condition for the bifurcation from a fixed point to a periodic solution.

Among others, analytical investigations were made mainly by [MEYER 1970], [MEYER 1971], [MIL'SHTEIN 1977], [BOTTKOL 1977] and [MÜLLER 1981]. MEYER investigated the bifurcation of periodic points (fixed points) of a vector field on a two-dimensional manifold which depends on one parameter. Based on these results, he classified the periodic points. Furthermore, he analysed the stability properties of each of these classes by the invariant curve-theorem of MOSER. By doing this he used stability criteria based on LIAPUNOV–functions. Also based on LIAPUNOV–functions, however in connection with optimization techniques, MIL'SHTEIN obtains criteria for the asymptotic stability of periodic solutions. MÜLLER modifies these stability theorems to apply them to limit cycles which are computed approximatively. However, he restricted his investigations to point-symmetric vector fields. BOTTKOL deals with vector fields, for which a parameter-dependent submanifold exists in the state space. His problem formulation concerns structural stability. That means, he investigates a vector field in the "neighborhood" and looks for a neighboring parameter-dependent submanifold of periodic solutions.

The investigation of bifurcation and stability of periodic solutions, which are numerically computed, can especially be found in [SEYDEL 1988].

1.2.4 Periodic solutions of dynamical systems with discontinuities

Already [SENATOR 1969] investigated stability and bifurcation aspects of periodic solutions with a one degree of freedom system subjected to impacts. Similar examples can be found in [HSU 1977] and [HARTOG 1931]. Since there is an analytical solution of these examples, the investigation is mainly focused on the special situation.

In the last few years, the number of authors dealing with numerical analysis of nonlinear systems with discontinuities has risen. Here particularly [HOLMES 1982], [SHAW 1985], [HEIMANN, BAJAJ, SHERMAN 1988], [PFEIFFER 1988a,b], [KARAGIANNIS 1989] and [MEIJAARD, DE PATER 1989] have to be mentioned. In these works the mathematical modelling, numerical computation and simulation are considered. Some authors such as HEIMANN et al. and SHAW also discuss the stability and bifurcation behavior of periodic solutions with discontinuities.

2 Differentiable dynamical systems

"The study of these cyclic or periodic vibrations is the study of vibration and it is one of the most important aspects of dynamics."

R.F. Steidel [2]

2.1 Preliminary remarks

The frame for the investigation – that is the numerical computation, stability and bifurcation – of singularities and periodic solutions are differentiable vector fields

$$\mathbf{f}^* : TM \times U \times P \times I \to TM \tag{2.1}$$

whose trajectory $\Phi_\xi : [0, \infty[\; \to TM$ with the initial point $\boldsymbol{\xi} \in TM$ is given for each time t by the unique initial value problem

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}^*(\mathbf{x}(t),\ \mathbf{u}(t),\ \mathbf{p},\ t)\ , \\ \mathbf{x}(0) &:= \boldsymbol{\xi}\ . \end{aligned} \tag{2.2}$$

In this formulation

M : is the configuration space (differentiable manifold, locally isomorphic to $\mathbb{R}^f$), $\dim M = f \in \mathbb{N}$, f is the number of degrees of freedom of the system,

TM : $= M \times \mathbb{R}^f$ is the state space,

$U \subset \mathbb{R}^m$: is the range of values of the control $\mathbf{u}$, which is mostly obtained by an optimization strategy or a control design,

$P \subset \mathbb{R}^k$: is the space of parameters, which can be varied in the system,

$I \subset \mathbb{R}$: is the range of time of excitation, which is mostly given by an explicit time function.

[2] From [STEIDEL 1989], page 40.

The vector field $\mathbf{f}^*$ is assumed to be sufficiently differentiable. The partial differentiability is related to the variables $\mathbf{x}, \mathbf{u}, \mathbf{p}$ and t.

∇

Example 2.1: Railway vehicle system.

Fig. 2.1 shows a simple dynamical model of a wheel set of a railway-vehicle system. The (rigid) wheel set (double cone with cone-angle δ) is elastically mounted in the undercarriage. The number of degrees of freedom is 6.

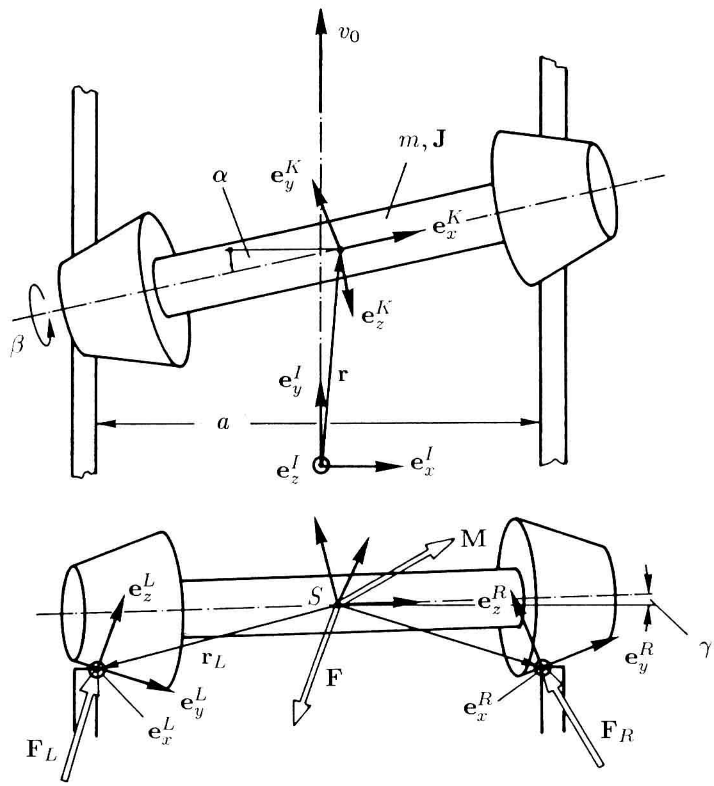

Fig. 2.1: Wheel set modelled as a double conus rolling on rectangular shaped rail

To investigate the dynamical behavior of the wheel set we assume that the undercarriage runs straight forward at a constant velocity $\mathbf{v}$. The feedback from the wheel set to the undercarriage is neglected because of the higher mass of the undercarriage. The position of the wheel set (with respect to an inertial frame $I := \left(\mathbf{e}_x^I, \mathbf{e}_y^I, \mathbf{e}_z^I\right)$, which is moved at a constant velocity v_0 of the wheel suspension) is described by the coordinates x, y and z as well as by the cardanian angles α, β and γ. Hence, the configuration space is given by

$$M = \mathbb{R}^3 \times SO(3) \ .$$